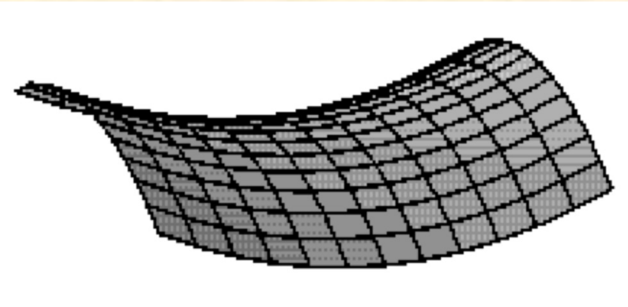
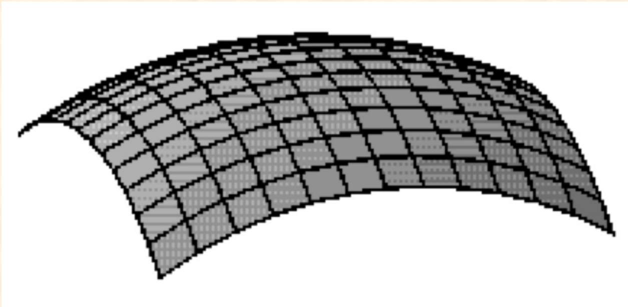
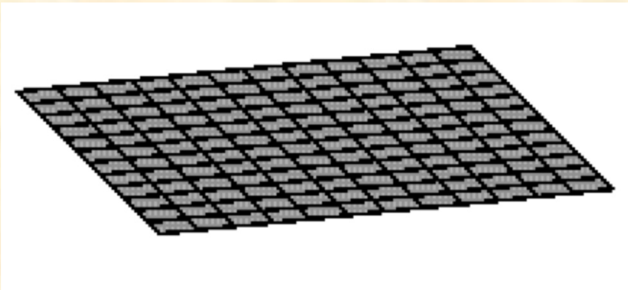
The Expanding Universe





- Metrics
- Redshift Corrections
- Distance Measures
- Look-back Time



Metrics

For a flat Universe (Euclidean geometry):

Minkowski metric of special relativity:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

In spherical coordinates:

$$ds^2 = c^2 dt^2 - dr^2 - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right)$$

Curved Universe

Cosmological Principle:

- The Universe is homogenous on scales $\geq$ 100 Mpc.
- The Universe is isotropic in all directions for every observer

For a curved expanding universe:

Robertson - Walker metric:

$$ds^2 = c^2 dt^2 - \frac{a(t)^2 dr^2}{\left(1 - kr^2/R^2\right)} - a(t)^2 r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

where $k = 0, \pm 1$ (sense of curvature)

R = radius of curvature

r, θ, ϕ = comoving coordinates (expand with the Universe)

a(t) = expansion parameter

A photon follows a "null geodesic" $\rightarrow$ $ds^2 = 0$

A galaxy at (r_1, θ_1, ϕ_1) emits a photon at time t_1 in the $-r$ direction, which arrives here at current time t_0. The photon's path is described by:

$$c\ dt = \frac{a(t)dr}{\left(1 - kr^2/R^2\right)^{1/2}}$$

$$c\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_0^{r_1} \frac{dr}{\left(1 - kr^2/R^2\right)^{1/2}} \equiv f(r_1) = \begin{cases} R\sin^{-1}\left(\frac{r_1}{R}\right) & \text{for } k = +1 \\ r_1 & \text{for } k = 0 \\ R\sinh^{-1}\left(\frac{r_1}{R}\right) & \text{for } k = -1 \end{cases}$$

For a successive wavefront emitted at time $t_1 + \delta t_1$:

$$c\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{a(t)} = \int_0^{r_1} \frac{dr}{\left(1 - kr^2/R^2\right)^{1/2}} = f(r_1)$$

Thus $\frac{\delta t_1}{a(t_1)} = \frac{\delta t_0}{a(t_0)} \quad \rightarrow \quad \frac{\delta t_0}{\delta t_1} = \frac{a(t_0)}{a(t_1)}$ (see Peterson, p. 139)

Since $\delta t \propto 1/\nu = \lambda/c$: $\rightarrow$ $\boxed{1 + z = \frac{\lambda_0}{\lambda_1} = \frac{a(t_0)}{a(t_1)}}$

Photons are redshifted by ratio of current size of Universe to that at time t_1

Redshift Transformations

$dt_0 = dt_1(1+z)$ (arrival of photons delayed by time dilation)

$\nu_0 = \nu_1/(1+z)$ (photons are redshifted)

The total number of photons in the emitted volume at frequencies ν_1 to $\nu_1 + d\nu_1$:

$n_{\nu 1}\ dA\ c\ dt_1\ d\nu_1$ (where $n_{\nu 1} =$ number density of emitted photons per Hz)

To conserve the number of photons in the box:

$$n_{\nu 1}\ dA\ d\nu_1\ c\ dt_1 = n_{\nu 0}\ dA\ d\nu_0\ c\ dt_0$$

$$= n_{\nu 0}\ dA\ \frac{d\nu_1}{(1+z)}\ c\ dt_1(1+z)$$

$$= n_{\nu 0}\ dA\ d\nu_1\ c\ dt_1$$

So: $\boxed{n_{\nu 1} = n_{\nu 0}}$

Photon flux (# of photons $\mathrm{s^{-1}\ cm^{-2}\ Hz^{-1}}$) = $n_{\nu 1} c$ ($\neq n_{\nu 1}$, Peterson, p. 152)

1) As previously shown: $\lambda_0 = \lambda_1(1+z),\ \nu_0 = \nu_1 / (1+z)$

2) Specific flux per frequency (ergs s^{-1} cm^{-2} Hz^{-1}):

$$F_{\nu 0} = h\nu_0 c n_{\nu 0} = h\frac{\nu_1}{1+z} c n_{\nu 1}$$

$$F_{\nu 0} = \frac{F_{\nu 1}}{1+z}$$

3) Integrated flux (ergs s^{-1} cm^{-2}):

$$F_0 = \int F_{\nu 0} d\nu_0 = \int \frac{F_{\nu 1}}{(1+z)} \frac{d\nu_1}{(1+z)}$$

$$F_0 = \frac{F_1}{(1+z)^2}$$ (due to decreased energy/photon plus decreased arrival rate)

4) Specific flux per wavelength (ergs s^{-1} cm^{-2} $Å^{-1}$):

$$F_{\lambda 0} = F_{\nu 0}\frac{c}{\lambda_0^2} = \frac{F_{\nu 1}}{(1+z)} \ \frac{c}{\lambda_1^2(1+z)^2}$$

$$F_{\lambda 0} = \frac{F_{\lambda 1}}{(1+z)^3}$$

5) Equivalent Width (Å) - transforms like λ: $W_{\lambda 0} = W_{\lambda 1}(1+z)$

Hubble Constant and Deceleration Parameter

Expand the expansion parameter with a Taylor expansion:

$$a(t) = a_0(t_0) + \dot{a}(t_0)(t - t_0) + 1/2\,\ddot{a}(t_0)(t - t_0)^2 + \ldots$$

$$a(t) = a_0\left[1 + \frac{\dot{a}(t_0)}{a_0}(t - t_0) + \frac{\ddot{a}(t_0)}{2a_0}(t - t_0)^2 + \ldots\right]$$

$$H_0 = \frac{\dot{a}(t_0)}{a_0} \qquad q \equiv -\frac{\ddot{a}\,a}{\dot{a}^2} \text{ (deceleration parameter)}$$

$$q_0 H_0^2 = -\frac{\ddot{a}\,a}{\dot{a}^2}\frac{\dot{a}^2}{a^2}\bigg|_{t_0} = \frac{\ddot{a}_0}{a_0}$$

$$a(t) = a_0\left[1 + H_0(t - t_0) - 1/2\,q_0 H_0^2 (t - t_0)^2 + \ldots\right]$$

If $q_0 > 0 \;\rightarrow\;$ deceleration

Critical Density and Cosmic Density Parameter

Combining the Robertson-Walker metric with the Einstein field equations and assuming a homogeneous Universe $\rightarrow$ Friedmann-Lemaître equations of motion

Assume no internal pressure (radiation or gas) and cosmological constant $\Lambda = 0$:

$$\frac{8\pi G\rho}{3} = H^2 + \frac{kc^2}{R^2 a^2} = 2qH^2$$ (see Peterson, p. 141 - 145)

For k = 0 and at the current epoch:

$$\rho_{\mathrm{crit}} = \frac{3H_0^2}{8\pi G} = 1.9 \times 10^{-29}\ h_0^2\ \mathrm{g\ cm^{-3}}$$

and $q = 1/2 \quad (q > 1/2 \text{ for } k = +1,\ q < 1/2 \text{ for } k = -1)$

Let $\Omega_0 \equiv \frac{\rho_0}{\rho_{\mathrm{crit}}} = \frac{\rho_0}{3H_0^2 / 8\pi G} = 2q_0$

(In general, $\Omega_0 = \Omega_{matter} + \Omega_{vacuum} = 2q_0 + \Lambda / 2H_0^2$ for a flat Universe)

Distance Measures

1) Proper distance (d_p): "real" distance between two galaxies at a fixed time

$$d_p = \int ds = \int_0^{r_1} \frac{a(t)dr}{\left(1 - kr^2/R^2\right)^{1/2}} = a(t_0) f(r_1)$$

$$\dot{d}_p = \dot{a}(t_0) f(r_1) = \dot{a}(t_0)\left(\frac{d_p}{a(t_0)}\right) = H_0 d_p \quad \text{(Hubble law)}$$

So: d_p is the distance to use for the Hubble law.

2) Angular size distance (d_a): distance "expected" from angular size:

The linear size of an object at distance r_1 is: $D = ds = a(t_1) r_1 \delta\theta$

$$d_a = D / \delta\theta = a(t_1) r_1$$

$$\frac{d_p}{d_a} = \frac{a(t_0) f(r_1)}{a(t_1) r_1} = (1+z)\frac{f(r_1)}{r_1}$$

So: Solve the above to get d_a

(Note: $d_a = d_p / (1+z)$ for $r_1 \ll R$.

Distance Measures

3) Luminosity distance (d_l): distance that corresponds to the observed flux via:

$$F_0 = \frac{L}{4\pi d_l^2}$$

In the reference frame of the galaxy:

$$F_1 = \frac{L}{4\pi (a_0 r_1)^2}$$

$$F_0 = \frac{F_1}{(1+z)^2} = \frac{L}{4\pi (1+z)^2 (a_0 r_1)^2} = \frac{L}{4\pi d_l^2}$$

$$d_l = (1+z) a_0 r_1 \quad (= a_1 r_1)$$

"It can be shown that":

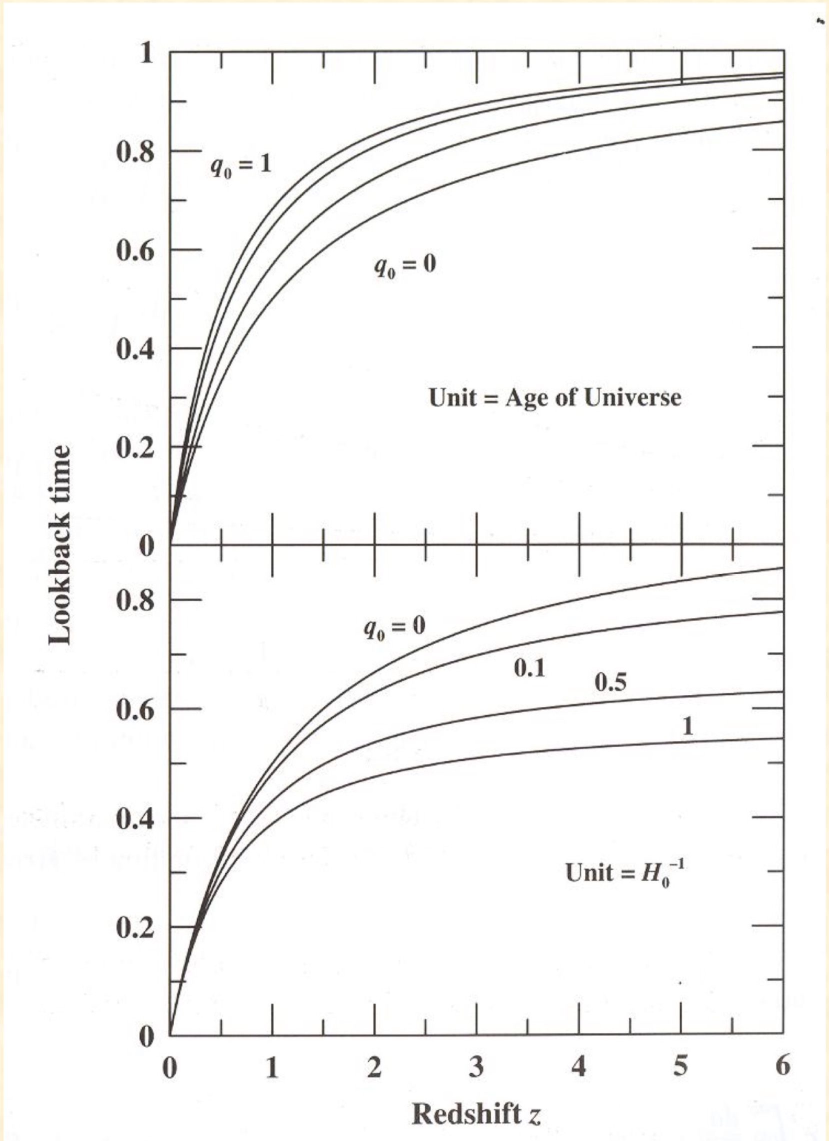
$$d_l = \frac{cz}{H_0}\left[1 + \frac{z(1-q_0)}{\sqrt{2q_0 z + 1} + 1 + q_0 z}\right]$$ (for a non-accel. universe, Peterson, p. 155)

So to get the luminosity: $L = F_0 4\pi d_l^2$

Lookback Times

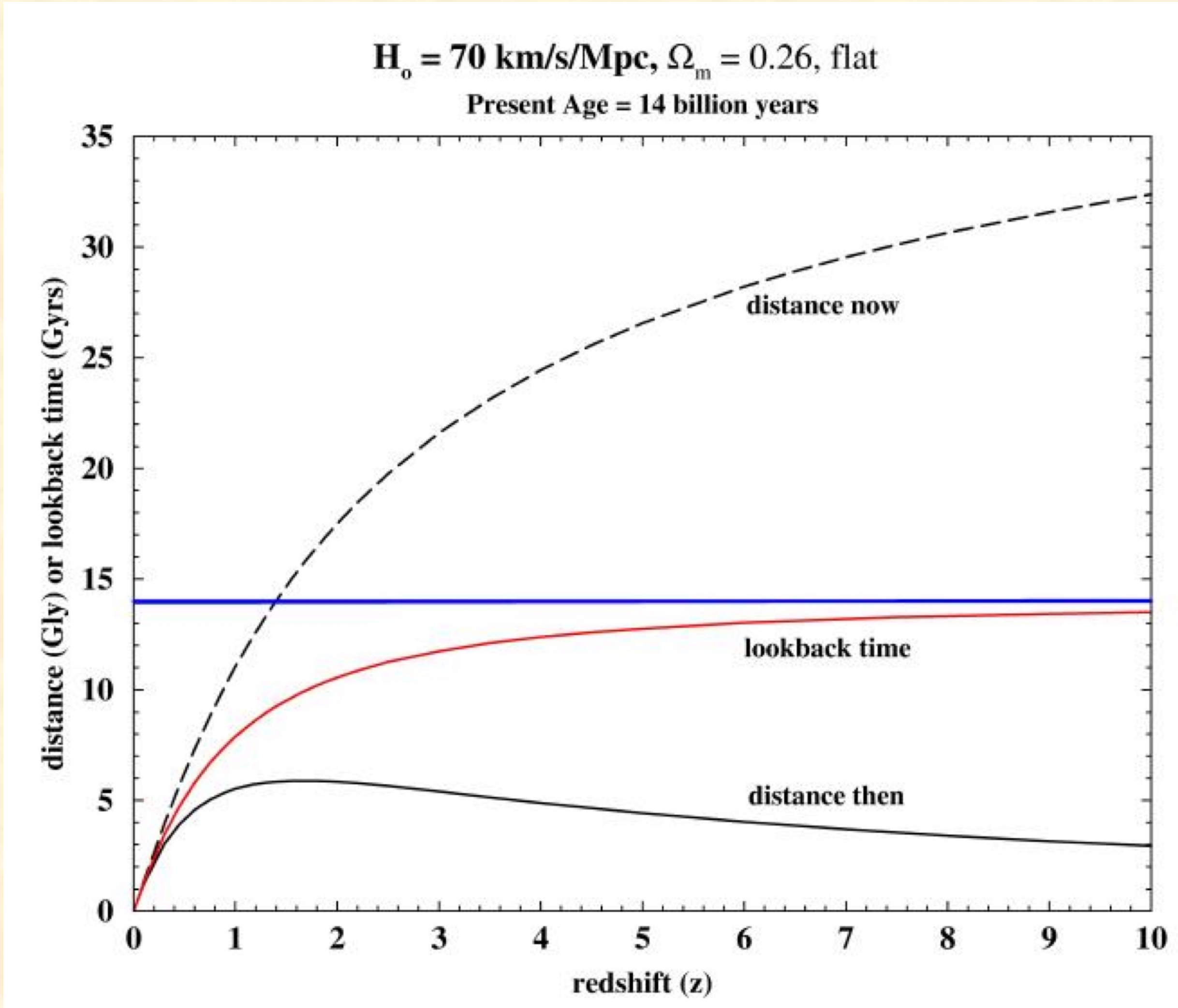
From Friedmann-Lemaître Equations and $\Lambda = 0$
(see Peterson, p. 145 - 151) $\rightarrow$ lookback time (τ)

Time that it takes for a photon emitted at z to reach us



(Peterson, p. 150)

Lookback Time for Accelerating Universe



http://homepages.wmich.edu/~korista/cosmology.html

Note - A very useful cosmological calculator can be found at:
http://www.astro.ucla.edu/~wright/CosmoCalc.html