Neutron Stars & Pulsars

Equation of State: "Intermediate density" from 
\(\rho_{\text{nuc}} = 2.8 \times 10^{14} \text{ g cm}^{-3}\) is probably well modeled by Baym-Bethe-Pethak EoS. Nuclei, e's & free \(n\)'s all contribute, but as \(\rho\) nuclei get more & more n rich, nuclei's stability \(\nabla \rho \to 0\) dissolve \(\to\) whole star is a giant nucleus.

Impressions on SEMF of HW:
1) Matter inside nuclei \& free n gas outside ... "Compressible liquid drop" assumed.
2) Surface energy includes free \(n\)'s \(\Rightarrow\) reduced surface energy term.
3) Nuclear lattice Coulomb energy included \(\nabla\) properly.

Above \(\approx 5 \times 10^{14} \text{ g cm}^{-3}\) can't try to use liquid drop ... \(\nabla\) but even before must try to incorporated NUCLEAR FORCE

Unfortunately, nuclear forces are non-linear \& don't obey superposition. \(\Rightarrow\) 3-body \& multi-body effects enter. But for \(\rho \approx \rho_{\text{nuc}}\) 2-body forces do dominate. Experiments \(\Rightarrow\) force invariant w.r.t. rotations, inversions \& time reversal; also not very \(\nabla\) dependent \(\Rightarrow\) \(\nabla \to 0\) \(\nabla\)

\[
V = V_1(r) + V_2(r)(\vec{\sigma}_1 \cdot \vec{\sigma}_2) + V_3(r)\left[3(\vec{\sigma}_1 \cdot \vec{n})(\vec{\sigma}_2 \cdot \vec{n}) - \vec{\sigma}_1 \cdot \vec{\sigma}_2\right]
\]

\(\vec{\sigma}_i = (\sigma_x, \sigma_y, \sigma_z)\), \(\sigma_x = (0,1), \sigma_y = (0,-1), \sigma_z = (1,0)\)

\(\sigma \cdot \sigma = 2\). \(\vec{\sigma}^2 = 2\).

\(\sigma\) are Pauli spin matrices.

But additional exchange interactions changing isospin.

\(T = 0 \Rightarrow T_3 = 0\): np only, \(T = 1\) (symmetric) \(\Rightarrow T_3 = \pm 1\)

\(\forall \) np, \(\forall n\).
All this $\Rightarrow$ force must saturate $\Rightarrow$ energy & volume $\propto A$. (Analogous to $2H \Rightarrow H_2$ but $H_3$ not stable). Nuclear force based on 3 aspects: 1) Exchange forces 2) Pauli exclusion principle 3) Repulsive core of the potential.

Above $10^{15}$ g cm$^{-3}$ relativistic effects & resonances are important - more uncertainty, but less important.

Simple form of nucleon-nucleon interaction:

Attractive outer part & repulsive core: Fix params. from N-N scattering & other experiments on compressibility, saturation energy, symmetry energy.

Examples: Reid pot is "soft" or very attractive.

Betho & Johnson, Welecko, Pandharipande & Smith have "stiffer" EoS w/ repulsive part dominant for $p > \rho_{\text{ave}}$.

Stiffer EoS $\Rightarrow$ 1) More support against collapse: $M_{\text{max}}$, $a)$ $p_c \downarrow$ $b)$ $\rho_c \uparrow$ $c)$ crust thicker. We hope that data on NAs will constrain models in ways nuclear physics experiments can't.

**Interaction Energy \textbf{\textit{Yukawa potential}}**

$$V_{1,2} = \frac{\pm g^2 e^{-r/\lambda}}{r} \quad \lambda = \text{Compton wavelength} \quad \text{of field quanta.}$$

$\pm$ : repulsive, vector; $-$ attractive, scalar

General form $\phi = \frac{\pm g^2 e^{-r/m}}{r} \quad (m = \frac{1}{2}) \quad (1)$

Note: $\lambda = 1.4 \, \text{fm} \Rightarrow m_{\text{quanta}} = 140 \, \text{MeV} \Rightarrow \lambda = 137$ for $E_\text{cm}$

A STRONG FORCE: $8^{-3} A \sim 5-20$ while $\frac{e^2}{\hbar c} = \frac{1}{\lambda}$ for $E_\text{cm}$
Simple analysis: \( E_{\text{tot}} = \sum E_{\text{pair}} \). Assume a uniform distribution, so ignore correlations from interactions.

For \( R \gg \lambda \):

\[
E_V = \frac{1}{2} \sum V_{ij} = \frac{1}{2} n^2 g^2 \int \frac{e^{-m r}}{r_{ij}} \, dV_i dV_j
\]

Pick origin \( O \) & integrate on shells. Ignore surf. effects \((R>>m)\).

\[
\int_0^\infty e^{-\frac{m r}{R}} \frac{4\pi r^2 \, dr}{r} = 4\pi \int_0^\infty e^{-\frac{m r}{R}} \frac{dr}{r} = \frac{4\pi}{m R}
\]

\[
E_V = \frac{1}{2} n^2 g^2 \pi^2 V \Rightarrow E = E_{\text{kin}} + 2n^2 g^2 \pi^2 V^2
\]

But:

\[
E_{\text{kin}} = \frac{n m c^2 + \frac{3}{10} \left( \frac{3 m^2}{5} \right)^{\frac{5}{3}} \pi^2}{\mu} n^5/3 \quad (NR)
\]

\[
= \frac{(3m)^{\frac{5}{3}}}{\pi^2} \, \frac{4\pi}{3} n^5/3 \quad (UR)
\]

Now: \( P = n^2 \frac{dE}{dn} \) so inc. both \( \text{kin} \) \& \( V \) terms:

\[
P = Kn^5/m^2 + 2n^2 g^2 \pi^2 V^2
\]

\begin{itemize}
  \item A.\, For \( P \)\, the attractive part of the nuclear force softens the eqn, speeding collapse. But later, the repulsive part enters, hardening the EoS.
  \item As \( n \to \infty \), \( P = E_{\text{kin}} \to \infty \) & \( n \)\, and \( P \to E = P c^2 \).
  \item But since: \( c_s = (\frac{dE}{dp})^{1/2} \), \( c_s \to c \) while for an ideal relativistic gas (e.g., photons) \( P \to \frac{E}{3} \Rightarrow c_s \to \sqrt{3} \).
\end{itemize}

This high \( c_s \) allows high total masses, but constraining \( c_s \leq c \) then, depending on \( m \), up to which you assume EoS is well known, Ruffini & Rhodes (1974) set from UML for \( M_T \) from \( 3-5M_0 \).

Interesting to note that lowest order QM approach (Heitler) exactly the same result, i.e., if spin exchange allowed: \( E = E_{\text{kin}} \pm \mu \nu g^2 \pi^2 R^2 \). Lower spin force repulsive, but raises for attractive force.
Improvements require including multi-particle wave functions via perturbation or cluster analysis. Most general is a variational technique, with assorted Yukawa-like terms.

\[ P = 364 \, \text{N} \, \text{m}^{-1} \, \text{MeV} \, \text{fm}^{-3} = 5.85 \times 10^{25} \, \text{N} \, \text{m}^{-1} \, \text{dyne} \, \text{cm}^{-2} \]

\[ C_s^2 = \frac{4P}{3\rho} = \frac{1}{1.01 + 0.568 \, \rho} \]

\[ \text{with} \, \rho \geq 1.54 \quad \text{for} \quad 0.1 \leq \rho \leq 3 \, \text{fm}^{-3} \]

\[ \text{Try to include } N, Z \text{ & } \Delta \text{ particles: net effect slight softens EoS -- probably ... but hard to be sure.} \]

Real problems involve:

1) \((\Delta, 1236 \, \text{MeV}, t = \frac{1}{2}, J = \frac{3}{2})\) resonance \(\Rightarrow\) stiffer EoS & \(n \rightarrow p + n^-\) when \(m_n < m_p < m_n \Rightarrow\) softer EoS & faster N\# cooling by \(\gamma\)'s.

2) Since \(\pi^-\)'s have \(S = 0\), are bosons, \(\ldots\) can condense \(\Rightarrow\) by softening via solidification of inner core of N\# \(\Rightarrow\) inner starquakes & superfluidity.

3) If \(p \approx \rho\) & relativistic potentials, poorly understood.

Hagedorn argued for asymptote: \(\text{resonance EoS} \Rightarrow P = \frac{\rho c^2}{\ln(\rho m_p)} \Rightarrow C_s^2 = \frac{c^2}{\ln(\rho m_p)} \left[ 1 - \frac{1}{\ln(\rho m_p)} \right] \]

\[ \Rightarrow \text{C}_s \rightarrow 0 \quad \text{as} \quad \rho m_p \rightarrow \infty \]

4) Quark matter \(\Rightarrow\) \(\text{dense,} \quad \text{if} \quad P \rightarrow \frac{1}{3} \rho c^2 \rightarrow \quad C_s \rightarrow \text{SOFT}. \quad \text{But this transition likely to occur above max P for stable N\#s.} \]

5) Recent alternative EoS (Bakerall and Neufeld Phys B31, 671) has hi \(\rho\) baryons bound by strong force, not grav \& objects more massive than N\# could be stable w/ \(P < P_{\text{crit}} \)

\[ \text{\# likely, just hypothetical, but if EoS needs} \] fusion, \(\text{alli} \Rightarrow \text{may weaken.} \)
Models of Neutron Stars

History: While proposed by Leendert, Gamow, Oppenheimer & Volkoff, astronomers ignored or ridiculed the idea, though thought of as source of Qasor redshift in 63. But in '67 pulsars found by J. Bell-Burnell under Ant. Hewish & T. Gold argued they were not. Not also found in most binary X-ray sources where accreted gas heats up in disk & on falling to polar caps of NS, funneled by B field.

Simplest model: Chandrasekhar-Nordheim $n = 3$
Polytrope allowing $p \rightarrow 0 \Rightarrow M_{ch} = 1.75 \, M_{\odot}$. But relativity lowers this, since max mass occurs at finite $p_c = M_{NS}$ just slightly rel. Bigger effect: $M_{ch}$ is rest mass of $M_{NS}$, but total mass less since $\beta$ grow, binding energy which reduces this.

Oppenheimer & Volkoff used TOV eqn & ideal n Eos to get $M_{max} = 0.7 \, M_{\odot}$, $R = 9.6 \, km$, $\rho_c = 5 \times 10^{15} \, g \, cm^{-3}$
Obtain approx model via $n = \frac{3}{2}$ (NR) polytrope:

$$R = 14.64 \left( \frac{\rho_c}{10^{15}} \right)^{-1/6} \, km \quad \& \quad M = 1.102 \left( \frac{\rho_c}{10^{15}} \right)^{4/3} \, M_{\odot}$$
with no minimum $M_{NS}$ mass, but this ignores regular $\beta$-decay as $A \rightarrow 0$. 


Energy Considerations:

\[ E = E_{\text{int}} + E_{\text{grav}} + \Delta E_{\text{int}} + \Delta E_{\text{gr}} \]

One finds \( \Delta E_{\text{gr}} \) by demanding \( \frac{\partial E}{\partial \rho_c} = 0 \) for polytrope

\[ (8) \quad E_{\text{int}} = \int \rho \text{d}m = \int \frac{n}{\nu^2} \text{d}m = K \gamma_{\text{c}} M \nu_{\text{c}} \int \frac{5}{3} \frac{r^{\frac{2n}{3}}}{r^{n-1}} \text{d}r \]

\[ \frac{\text{grav}}{} = -G \int \frac{\nu^2}{m} \text{d}m = -\frac{3}{5-n} \frac{GM^2}{R} \]

Using an \( n = \frac{7}{2} \) polytrope for low \( \rho \), \( \nu_{\text{c}} \approx \rho_{\text{c}} \)

\[ E_{\text{int}} = k_1 K \rho_{\text{c}}^{4/3}, \quad k_1 = 0.795873 \]

\[ E_{\text{grav}} = -k_2 G \rho_{\text{c}}^{4/3} M^{5/3}, \quad k_2 = 0.760727 \]

\[ \Delta E_{\text{int}} \] arises from change of state from NR deg. gas

The internal energy/unit mass is:

\[ U = \frac{E_{\text{int}} - M \nu_{\text{c}}^2}{M_{\text{c}}} \]

\[ P_0 = \frac{M \nu_{\text{c}}^2}{3 \pi^2 R_{\text{c}}^3} \]

\[ U = C^2 \left( \frac{3}{10} x - \frac{5}{36} x^2 + \ldots \right) \]

Integrating \( R \Rightarrow \int E_{\text{int}} \Rightarrow \Delta E_{\text{int}} \)

\[ \Delta E_{\text{int}} = -\frac{3}{5} \frac{C^2}{2} \int x^2 \text{d}m = \gamma_{\text{c}} \frac{9}{2} \left( \frac{x}{\nu_{\text{c}}} \right)^4 \frac{M_{\text{c}}^{4/3}}{M_{\text{c}}} \]

\[ \gamma_{\text{c}} \approx \left( \frac{5/3}{5-1} \right) \frac{1}{2} \int \frac{5}{3} \frac{r^{2/3}}{r^{n-1}} \text{d}r = 1.165 \]

The final term in (7), \( \Delta E_{\text{gr}} \), is quite complex. Note

\[ dU = (1 - 2\mu)^{1/2} 4\pi r^2 \text{d}m \]

\[ E_{\text{gr}} = \int \frac{\rho \text{d}m}{r^2} \left( \frac{1 - 2\mu}{r} - \mu \right) \text{d}V \]

\[ \rho = \text{rest mass density} \quad \mu = \frac{\rho \text{c}}{1 + \nu} \]

\[ E_{\text{newt}} = \int \frac{\rho \text{d}m}{r^2} \left( \frac{1 - 2\mu}{r} - \mu \right) \text{d}V \]

take off binding energy!

\[ \Delta E_{\text{gr}} = -k_4 \frac{G^2}{c^2} M_{\text{c}}^{7/3} \frac{2}{15} \left( \frac{5/2 \pi}{\nu_{\text{c}}} \right)^{2} \frac{5/2}{2} \text{d}r^2 + \frac{3}{7} (r-u) \left( \frac{5/2 \pi}{\nu_{\text{c}}} \right)^{2} \frac{5/2}{2} \text{d}r^2 \]

where \( k_4 \approx 0.639725 \)
Finally, the total energy is:

\[ E = AM\rho_c^{2/3} - BM\rho_c^{5/3} - CM\rho_c^{1/3} - DM\rho_c^{7/3} \]

with:
\[ A = k_1 K, \quad B = k_2 G, \quad C = \frac{k_3}{\mu_0 n^2 c^2}, \quad D = \frac{k_4}{c^2} \]

Equilib\(\nu\) @ \( \frac{dE}{d\rho_c} = 0 \), or, multiplying by \( \frac{3}{2} \mu_0 \):

\[ 2A\rho_c^{-1/3} - BM\rho_c^{2/3} - 4C\rho_c^{1/3} - 2D\rho_c^{4/3} = 0 \]

Using only \( A \) & \( B \) terms results (6) for Newton polytopes.

Get maximum mass when energy unstable: \( \frac{d^2E}{d\rho_c^2} < 0 \)

Multiply by \( \frac{3}{2} \mu_0 \) to get:

\[ -A\rho_c^{-1/3} + BM\rho_c^{2/3} - 2C\rho_c^{1/3} + DM\rho_c^{4/3} = 0 \]

Solve by adding (16) + 2x(17) to get:

\[ \rho_c = \frac{BM^{2/3}}{8C} \]

Plug into (16) & let \( y = M^{4/9} \) to get:

\[ 2A - 3BM^{2/3} C^{4/9} y - 2Dy^3 = 0 \]

(18) Has positive root @ \( y = 6.605 \times 10^{-4} \Rightarrow M_{\text{max}} = 1.11 M_\odot \)

Plug \( M \) & \( \rho_c \) into (15) to get \( \frac{E}{c^2} = -0.08 M_\odot \Rightarrow \text{total mass} \):

\[ M_{\text{poly, grav}} = 1.03 M_\odot \]

However, full, non-polytopic TOV \( \Rightarrow M = 0.7 M_\odot \)

An improvement might be n-e-p plasma:

\( M_{\text{max}} = 0.12 M_\odot, \quad R_{\text{max}} = 8.8 \text{ km}, \quad \rho_{\text{c}} = 5.8 \times 10^{15} \)

HW showed \( \exists \) minimum mass too:

\( M_{\text{min}} = 0.18 M_\odot, \quad R = 300 \text{ km}, \quad \rho_{\text{c}} = 2.6 \times 10^{13} \)
Improved EOS $\Rightarrow$ better estimates of conditions.

$\text{Soft-based just on TT & Reid potential } \Rightarrow M_{\text{max}} = 1.5 - 1.6$

$\& \ R_{\text{max}} = 8\text{ km}$

$\text{Hard - Bethe-Johnson w/tensor & 3N interactions } \Rightarrow M_{\text{max}} = 1.9 - 2.2$

$\& \ R_{\text{max}} = 10 - 15\text{ km}$

$\text{Stiffest - } \text{"Mean field" approx: } M_{\text{max}} = 2.7 M_{\odot} \text{ (unlikely).}$

**Interior Structure of NS's**

1. **Surface** $\rho \lesssim 10^6 \text{ g/cm}^3$ thick. Strong B fields & $\nabla T$ allow diff EOS to affect cooling rate.

2. **Outer crust:** $10^6 \leq \rho \leq 4.3 \times 10^7 \text{ g/cm}^3$ 0.1 - 0.5 km thick.

   Like a WD solid crust - Coulomb lattice of heavy nuclei in $\beta \text{-} w$ rel. degem. e-gas

3. **Inner crust:** $4.3 \times 10^7 \leq \rho \leq 2.2 \times 10^9 \text{ g/cm}^3$ ~ 2 - 8 km

   Neutrin rich nuclei & superfluid neutrino gas $\&$ e-gas

4. **Neutron liquid:** $2.2 \times 10^9 \leq \rho \leq \text{ Proce [dep. on EOS]}$

   ~ 8 - 12 km. Superfluid $n$'s w/ some superfluid $p$'s & normal $e$'s. Most of the NS

5. **Core [in some models] $\rho \gtrsim 10^{15}$ where

   $\text{n condensation, solid } n$'s $\&$ for quarks may live.

   But for most EOS $\rho_c \lesssim 3 \times 10^{15}$ = $\text{quarks unlikely}$

   **Minimum mass:** Where $J^2 > 4\hbar^2$. Around ~ $7 \times 10^{15}$ g

   Ultra well determined, so:

   $M_{\text{min}} = 0.093 M_{\odot}, \rho_c = 1.55 \times 10^9, R = 184 \text{ km}$

   But such low mass objects have no clear antecedent.
Maximum Mass: (Rhodes & Russia 1974) Since $P_0 > P_{	ext{max}}$ isn't well known, neither is this. Absolute max by assuming:

1. GR is true — use TOV eqns.
2. Microscopic stability: $\frac{\partial P}{\partial \rho} > 0 \Rightarrow$ matter won't collapse
3. Causality $\frac{\partial P}{\partial \rho} \leq \frac{c}{c}$, i.e. $c_s \leq c$
4. EOS @ $\rho \leq \rho_0$ is known well enough. If $\rho_0 \sim 10^{10} \text{g/cm}^3$

$$M_{\text{max}} \approx 3.2 \left(\frac{\rho_0}{10^{15} \text{g/cm}^3}\right)^{-1/2} M_{\odot}$$

causality: $M_{\text{max}} \approx 5.2 \left(\frac{\rho_0}{10^{15} \text{g/cm}^3}\right)^{1/2} M_{\odot}$

N.B. Rapid rotation probably allows at most 20% increase before unstable.

**Observed Properties of Neuts**

1. Measured Masses:
   a) PSR 1913+16 is single-line "radio binary. But precession of perihelion & transverse Doppler shift $M_{\text{psr}} - M_{\text{com}} = 1.41 \pm 0.03 M_{\odot}$. Other binary pulsars w/ mass estimates — all consistent w/ this.
   b) Pulsating X-ray binaries $\Rightarrow$ mass fnw w/ nominal values from 1.1 to 2.31$^{+2}_{-1}$ — all consistent w/ 1.4-1.6 $M_{\odot}$. Recall — this is range expected from pool of w/ MS mass up to 130 $M_{\odot}$ t/or collapse of WD over $M_{\odot}$.

2. Measured Radii: None good. Models of X-ray bursters using $\Delta X$ on Fe lines. $R \approx 8.5 \text{ km}$ almost certain. e$^+$$e^-$ annihilation line results also consistent w/ 10-12 km.