Parallel Transport for Tensors & Vectors

For vectors, if $A^i @ P(x^i) & Q(x^i + \delta x^i)$ is neighbouring, how do we shift $A^i$ from $P$ to $Q$ without changing its magnitude or direction?

$$\delta A^i = -\Gamma^i_{jk} A^i \delta x^j$$

(1)

Affine connection of $T$-T or Christoffel symbols.

If $\delta$ is given by $x^i(\lambda)$ so that $A^i @ X_\lambda$ on $\lambda & \lambda_1$, when transported $\parallel$ to itself is:

$$\frac{dA^i}{d\lambda} = -\Gamma^i_{jk} A^k \frac{dx^j}{d\lambda}$$

(if $\frac{dx^j}{d\lambda}$ exists) (2)

For covariant components, since a scalar shouldn't change under $\parallel$ trans:

$$0 = \delta(A^i B_i) = \delta A^i B_i + A^i \delta B_i =$$

$$-\Gamma^i_{jk} A^k \delta x^j B_i + A^i \delta B_i$$

e.g. $A^k(\delta B_k - \Gamma^i_{kl} B_i \delta x^l) = 0 \rightarrow$

$$\delta B_k = \Gamma^i_{kl} B_i \delta x^l$$

(3) (as $A^i B_i$ are arbitrary)

N.B. Since $A_i : A^i$ is unchanged under $\parallel$ trans, the magnitude of $A$ is preserved.
Generalize to Tensors of Arbitrary Rank:

\[ \delta T_{ik} = \Gamma^l_{im} T_{lk} \delta x^m + \Gamma^l_{km} T_{il} \delta x^m \]  \tag{4} 

Covariant Differentiation

For a vector field \( A^i @ P(x^i) \& Q(x^i+\delta x^i) \): the genuine change isn't: \( dA^i = \frac{\partial A^i}{\partial x^0} \delta x^0 \) but by: 

\[ dA^i - \delta A^i = \left( \frac{\partial A^i}{\partial x^k} + \Gamma^i_{kl} A^l \right) \delta x^k \]

The genuine change should be coordinate independent, trans. like a vector. As \( \delta x^k \) is a contra vector:

\[ A^i_{,l} = \frac{\partial A^i}{\partial x^k} + \Gamma^i_{kl} A^k \equiv A^i_{,l} + \Gamma^i_{kl} A^k \] \tag{5}

The covariant derivative \( \cdot \cdot \) is a mixed tensor

Similarly: 

\[ \phi_{,l} = \phi_{,l} \]

\[ B_{,l} = B_{,l} - \Gamma^k_{il} B^k \]

\[ T_{ik,l} = T_{ik,l} - \Gamma^m_{ik} T_{lm} - \Gamma^m_{il} T_{km} \] \tag{6}

Transformation Law for \( \Gamma^i_{km} \): Let \( A^i \rightarrow A'^i \) as \( x^i \rightarrow x'^i \):

\[ A'^i = \frac{\partial x'^i}{\partial x^m} A^m \Rightarrow A'^i_{,l} = \frac{\partial x'^i}{\partial x^m} \frac{\partial x^m}{\partial x'^j} A'^j_{,l} \]

as covar. deriv. is mixed 2nd rank tensor.
But (5) + 1st relation on last line →

\[ A'_{jle} = \frac{\partial}{\partial x'^{l}} \left( \frac{\partial x'^{i}}{\partial x^{m}} A^{m} \right) + \Gamma'_{li} \quad \text{[\textit{r}]} \]

\[ = \frac{\partial}{\partial x^{n}} \left( \frac{\partial x'^{i}}{\partial x^{m}} A^{m} \right) \frac{\partial x^{n}}{\partial x'^{l}} + \Gamma'_{li} \frac{\partial x'^{p}}{\partial x^{m}} A^{m} \]

\[ = \frac{\partial x^{n}}{\partial x'^{l}} \frac{\partial x'^{i}}{\partial x^{m}} \frac{\partial A^{m}}{\partial x^{n}} + \left( \frac{\partial x^{n}}{\partial x'^{l}} \frac{\partial x'^{i}}{\partial x^{m}} + \Gamma'_{li} \frac{\partial x'^{p}}{\partial x^{m}} \right) A^{m} \]

However, 2nd rel. on last line of (25) →

\[ A'_{jle} = \frac{\partial x^{n}}{\partial x'^{l}} \frac{\partial x'^{i}}{\partial x^{m}} \left( \frac{\partial A^{m}}{\partial x^{n}} + \Gamma^{m}_{pn} A^{p} \right) \]

\[ = \frac{\partial x^{n}}{\partial x'^{l}} \frac{\partial x'^{i}}{\partial x^{m}} \frac{\partial A^{m}}{\partial x^{n}} + \frac{\partial x^{n}}{\partial x'^{l}} \frac{\partial x'^{i}}{\partial x^{m}} \Gamma^{m}_{pn} A^{p} \]

[switching dummy p & m]

Comparing both \( A'_{jle} \) & recalling \( A^{m} \) is arbitrary:

\[ \frac{\partial x^{n}}{\partial x'^{l}} \frac{\partial x'^{i}}{\partial x^{m}} \Gamma^{m}_{pn} = \frac{\partial x^{n}}{\partial x'^{l}} \Gamma^{m}_{pi} + \frac{\partial x^{n}}{\partial x'^{l}} \frac{\partial x'^{i}}{\partial x^{m}} \frac{\partial x^{m}}{\partial x^{n}} \]

Multiply both sides by \( \frac{\partial x'^{i}}{\partial x^{k}} \) & recall \( \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x^{k}}{\partial x'^{l}} = \delta^{i}_{l} \); then the above reduces to:

\[ \Gamma^{r}_{mk} = \frac{\partial x^{r}}{\partial x^{k}} \frac{\partial x'^{l}}{\partial x^{i}} \frac{\partial x^{p}}{\partial x^{m}} \Gamma^{m}_{pi} + \frac{\partial x^{r}}{\partial x^{k}} \frac{\partial x^{m}}{\partial x^{i}} \frac{\partial x^{p}}{\partial x^{m}} \]

\[ \text{[3rd rank tensor part]} \quad \text{[non-tensor part]} \]

i.e. Christoffel symbols aren't tensors.
Riemannian Affine Connection

\[ + 64 \Gamma_{ik}^i \text{. Impose the following:} \]
\[ \Gamma_{ik}^i = \Gamma_{ki}^i \quad \text{(8)} \]
\[ g_{i[k;j]l} = 0 \quad \text{(9)} \]

While other geometries can exist, GR is based on Riemannian geometry; (8) & (9) hold.

Use (9) & (6) to show: \[ \Gamma_{ik}^m g_{mk} + \Gamma_{ik}^m g_{mn} = g_{ik;l} \]

or \[ \Gamma^{[ik]} + \Gamma^{[i}] \Gamma^{[l]} = g_{ik;l} \quad \text{(10)} \]
where, using (8): \[ \Gamma^{[ik]} = g_{mk} \Gamma^{[i]} = \Gamma^{[k]} - \Gamma^{[i]} \]

Make cyclic interchanges on \( i, k, l \) to get, \( k, i, l \): \[ \Gamma_{k[i} + \Gamma_{i]k} = g_{k[i} \Gamma_{i]k} \quad \text{and} \]
\[ \Gamma_{ik}^m = \frac{1}{2} g_{m[l} \left[ g_{ik} + g_{ik} - g_{ik} \right] \quad \text{(13)} \]

Add (12a)+(12b); subtract (10), then get
\[ \Gamma_{ik} = \frac{1}{2} \left[ g_{ik} + g_{ik} - g_{ik} \right] \quad \text{and} \]
\[ \Gamma_{ik} - \frac{1}{2} g_{m[l} \left[ g_{ik} + g_{ik} - g_{ik} \right] \quad \text{(13)} \]

.: The Riemannian affine connection is completely specified by S-T metric & 1st derivatives; instead of 64 \( \rightarrow \) 40 algebraically independent components.

Use (13) & differential of determinant: \( dg = g^{-1} \text{d}g \text{ik} \) to show:
\[ \Gamma_{ik} = \frac{2}{2} \left[ \ln \sqrt{-g} \right] \quad \text{and} \]
\[ g_{ik}^{-1} \text{d}g_{ik} = - \frac{1}{\sqrt{-g}} \frac{2}{2} \left[ \ln \sqrt{-g} \right] \quad \text{and} \]
\[ g_{ik}^{-1} \text{d}g_{ik} = - \frac{1}{\sqrt{-g}} \frac{2}{2} \left[ \ln \sqrt{-g} \right] \quad \text{(13)} \]
Locally Inertial Coordinate System:

Since the metric at any point is assumed to have signature +2, it is evidently possible to choose coods \( x^i \rightarrow P \) such that:

\[
\text{ds}^2 = g_{ik} \, dx^i \, dx^k
\]

Given a Riemannian connection we can choose a coordinate system \( \{a \text{ LIC}S\} \)

\[
g_{ik}(P) = \delta_{ik}, \quad g_{ik,l}(P) = 0 \quad (14)
\]

Minkowskian metric holds in infinitesimal neigh. of \( P \).

Proof: Suppose in \( x^i \) the coods of \( P \) are \( x'^i \) & Christoffel symbols at \( P \) are \( \Gamma^{iv}_{klo} \). Now define, in a small region containing \( P \) new coods \( x^i \):

\[
x'^i - x'^i = x^i - \frac{1}{2} (\Gamma^{iv}_{klo})_P x^k x^l
\]

\( \Gamma^{iv}_{klo} \) vanish at \( P \) we have \( x^i = 0 \) &

\[
\frac{\partial x'^i}{\partial x^k} = \delta^i_k
\]

while \( \frac{\partial^2 x'^i}{\partial x^k \partial x^l} = (\Gamma^{iv}_{klo})_P \)

Plugging these into (7) we see that in the \( x^i \) coord system the \( \Gamma \)'s \( \text{at} \ P \) vanish. Then use (10) & \( g_{ik,l} = 0 \) for these coods. Finally diagonalize the line element \( \text{at} \ P \) to yield (14).

We'll discuss the importance & significance of the LICs later.
**Lie Derivative**

Let $A$ & $B$ be 2 vector fields & $f$ a scalar field. Define the covariant derivative of $f$ along $A$ by:

$$\nabla_A f = A^i f^i$$  \hspace{1cm} (16)

Similarly cov. deriv. $\nabla_A B$ of $B$ along $A$ is defined as a vector by components given by:

$$\left( \nabla_A B \right)^i = A^k B^i_{;k}$$  \hspace{1cm} (17)

Consider the quantity:

$$\nabla_{[B,A]} f = \nabla_B \nabla_A f - \nabla_A \nabla_B f = (B^i_{;i} A^i - A^i_{;i} B^i) f_{,k}$$

Similarly define:

$$\left[ \nabla_{[B,A]} C \right] = (B^i_{;i} A^i - A^i_{;i} B^i) C_{;k}$$

The operator $\nabla_{[B,A]}$ is a generalization of Lie derivative in Euclidean space. Define the commutator $[B,A]$ or the Lie derivative of $A$ along $B$ by:

$$[B,A] = \nabla_B A \rightarrow (B^i A^i_{;i} - A^i B^i_{;i})$$  \hspace{1cm} (18)

where $\rightarrow$ indicates $k^{th}$ component of LHS vector.

Obviously:

$$\nabla^k A = -\nabla_k B$$  \hspace{1cm} (19)

Even in Minkowski $S-T$

$$\bar{P} \bar{Q} = A \bar{Q}, \bar{Q} \bar{R} = B \bar{Q}$$

but $\bar{P} \bar{S} = B \bar{P} \bar{Q} \bar{T} \equiv A \bar{Q} \bar{S}$

In general, $R \neq T$ & diff is the Lie derivative.

In curved $S-T$ the interpretation is the same but can't draw straight lines; instead must draw tangents to curves for which $A$ & $B$ are tangent vectors.