Energy Momentum Tensors

1. Action Principle
2. E&M Stress Energy Tensor
3. Eu. Mom Tensors in General
4. Special Cases Used in GR

For example, in E&M conservation of energy (except where it charges) is expressed by:

\[ T^{\mu \nu} = 0 \]  \hspace{1cm} (1)

Where \( \phi \) charges, the RHS \( \neq 0 \Rightarrow \) eqns. of motion for charged particles. Since divergence is in eqn \( \Rightarrow \) free to add a div.-free tensor arbitrary. This is not allowed in GR, since, as we'll see, \( T^{\mu \nu} \) acts as a source for Einstein's eqns (matter \( \rightarrow S = T \)). Thus we must explicitly & systematically prescribe the energy-momentum (or stress-energy) tensor.

Action Principle (Hamilton, 1834)

A scalar function of generalized dynamical coordinates \( \bar{g}_r \) \((r = 1, 2, \ldots N)\), \( N \) = #d.o.f. & time derivatives \( \dot{g}_r \) & time \( t \):

\[ L = L(g_r, \dot{g}_r, t) \]  \hspace{1cm} (2)

The action integral depends on the 'path' \( g_r = g_r(t) \):

\[ A = \int_{t_1}^{t_2} L \, dt \]  \hspace{1cm} (3)

Actual motion is specified by \( g_r = \bar{g}_r(t) \Rightarrow A \) stationary:

\[ \delta A = 0 \]  \hspace{1cm} for \( g_r(t) = \bar{g}_r(t) + \delta g_r(t) \) \hspace{1cm} (4)

subject to end conditions specified at \( t_1, t_2 \). This \( \Rightarrow \) Lagrange's Eqns.
We've already done something like that for geodesics: these, as we'll discuss, give the motion of 'free test-particles' in $S^T$. But action principles aren't limited to dynamics; E&M & QED also can be derived this way.

- General action principle: suppose a system is described by a series of functions, $\phi^A (A=1, 2, \ldots)$ of $S^T$ coords $x^i$. From $\phi^A$ derive $\phi^A_i$, construct a Lagrangian density:
  \[ L = L (\phi^A, \phi^A_i, x^i) \]  
  (5) as a scalar function, & define the action integral as:
  \[ A = \int U L \, d^4x \]  
  (6)

where $U$ is the specified domain of $S^T$ manifold, w/ boundary surface $\Sigma$. Then, eqns satisfied by $\phi^A$ are:

(7) $\delta A = 0$, for small variations $\delta \phi^A (\phi^A = \phi^A + \delta \phi^A)$, $\delta \Sigma$.

We use the relation $\delta (\phi^A) = (\delta \phi^A)_i$, valid for infinitesimal variations. Sometimes we must generalize $\delta$ if interactions w/particles described by world-lines are needed, for then line integrals are involved. But line integrals can be transformed to vol. $S$ using 4-Dirac $\delta$; defined, for source pt. $A \in S$ & field pt. $X$:

(8) $\delta_X (X, A) = 0$ if $X \neq A$; $\int S \delta_X (X, A) = 1$.

Action principle is simple & it works. But it has deeper meaning. Dirac proposed & Feynman developed path integral approach. Say system described by (8) starts @ $I$ & proceeds to $F$. How many ways can it get there?

Let $S_{IF}$ be a path; $\exists$ many (usually $\infty$) such paths; for each, compute:

(9) $\mathcal{A} [S_{IF}] = \int \mathcal{E}^I e^{i S_{IF}} d^4x$ where each $S_{IF} \rightarrow$ specified path $\gamma (x)$.

The probability amplitude for passage $S_{IF}$ is:

$P(S_{IF}) = \propto \exp \left[ \frac{i \mathcal{A} [S_{IF}]}{\hbar} \right]$ 

The total prob. amp is:

$K (F; I) = \sum_{S_{IF}} P(S_{IF})$  

(10)
on a similar path-integral if paths form a continuum. (11) is the formal defn of Feynmann propagator & contains all QM behavior of system. As \( n \to 0 \) should \( \rightarrow \) classical limit. Then oscillates wildly & unless oscillations are systematically phased, \( \Sigma \) in (11) will be 0. Such phasing occurs only if values of \( A \) are the same, i.e. as \( n \to 0 \), only contributors have \( \delta A = 0 \). (7) describes the most probable, ideally certain, behavior of a system in classical physics.

**Electromagnetic Theory**

We can derive Maxwell-Lorentz electrodynamics eqns in Riemannian S-T using the action principle.

\[
\mathcal{A} = -\sum_{a} \int \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \, d^4x - \sum_{a} e_{a} A_{c} \, da^{c} \tag{12}
\]

*G\text{inear}t\text{erm}, all masses \( m_{a} \) w/\( da \) the element of proper-time along its world line

\( da^{2} = g_{\mu\nu} da^{\mu} da^{\nu} \) \tag{13}

pure field term

interaction of a \( a \)th particle, having charge \( e_{a} \), with EM field whose potential is \( A_{i} \)

\[
F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}
\]

\( \mu \neq \nu \), the covar dir has reduce to ord as is anti-sym.

Now we'll study \( \delta \mathcal{A} = 0 \) for 2 types of variations:

1. Keep \( A_{i} \) unchanged as \( f_{a} \) of \( x^{i} \) but vary world lines.
2. Fix particle world lines but change functional forms of \( A_{i} \)

For 1), then 2nd term in (12) contributes nothing.

1st term \( \rightarrow \) Euler-Lagrange expression for geodesics discussed earlier

\[
\left( m_{a} \frac{d^{2}a^{i}}{ds^{2}} + m_{a} \Gamma^{i}_{kl} \frac{da^{k}}{ds} \frac{da^{l}}{ds} \right) = 0
\]

So we now consider variations on the last term of (12) subject to fixed end pts, \( P_{1}, P_{2} \) on world line of \( a \):
\[-\delta \left[ e_a \int P_i^2 \, da \right] = -e_a \int P_i^2 \left\{ A_{i}^\text{e} \frac{\partial A_{i}^\text{e}}{\partial a} + A_{i} \frac{d}{da} (\delta a^e) \right\} da
\]
\[= -e_a \int P_i^2 \left\{ A_{i} \frac{da^i}{da} - A_{i} \frac{d}{da} \delta a^e \right\} da = -e_a \int P_i^2 \frac{d}{da} \delta a^e \, da.\]

Combining the 1st & 3rd term variations we get the Lorentz-force-eqn for motion of charged particles \(a_i\):

\[
M_a \left[ \frac{d^2 a^i}{d a^2} + \gamma \frac{d}{d a} \left( \frac{d a^i}{d a} \right) \right] = e_a F^i \, \frac{d a^k}{d a}
\]

For variation 2), consider the 4-volume \(V\) bounded by a 3-surface \(\Sigma\) over which the variations \(\delta A_i = 0\). Also,

\[
\delta F_{ik} = \delta A_{ik} - \delta A_{ki}
\]

Then using anti-sym of \(F_{ik}\), the 2\textsuperscript{nd} term gives:

\[
-\delta \left[ \frac{1}{16 \pi} \int \frac{1}{V} F_{ik} F^{ik} \, d^4 x \right] = -\frac{1}{16 \pi} \int \frac{1}{V} F^{ik} (\delta A_{ik} - \delta A_{ik}) F_{ik} \, d^4 x
\]
\[= -\frac{1}{16 \pi} \int \frac{1}{V} F^{ik} \delta A_{ik} F_{ik} \, d^4 x = \frac{1}{16 \pi} \int \frac{1}{V} F_{ik} \delta A_{ik} F^{ik} \, d^4 x
\]
\[= -\frac{1}{16 \pi} \int \frac{1}{V} F_{ij} \delta A_{ik} F^{ik} \, d^4 x + \frac{1}{16 \pi} \int \frac{1}{V} F_{ik} \delta A_{ik} F^{ik} \, d^4 x
\]

(For in general, for anti-sym tensors \(F^{ik}\), \(\frac{1}{V} \int \frac{1}{16 \pi} F^{ik} F_{ik}\) is.

But, as \(\delta A_i = 0\) on \(\Sigma\), we have:

\[
-\delta \left[ \frac{1}{16 \pi} \int \frac{1}{V} F_{ik} F^{ik} \, d^4 x \right] = \frac{1}{16 \pi} \int \frac{1}{V} F_{ik} \delta A_{ik} F^{ik} \, d^4 x
\]

The 1st term, w/ no \(E\&M\) part, is unchanged by \(\delta\). Consider a typical member, \(b\), of sum in 3\textsuperscript{rd} term, \(w/\) a pt on its world-line

\[
-\delta \left[ e_b \int S_{i} \, da \right] = -\delta \left[ e_b \int S_{i} \delta a^e (x) \, da \right]
\]
\[= -e_b \int S_{i} \delta a^e (x) \, da \, d^4 x = -\int S_{i} \delta a^e (x) \, d^4 x
\]

where the current density due to \(b\) is \(j_i\):

\[
j_i = e_b \int S_{i} \, da + \gamma \frac{d}{d a} \left( \frac{d a^i}{d a} \right)
\]

(19)
Add contributions of each charge to get total current density vector:
\[ \mathbf{j}^c(x) = \sum_{\delta} \epsilon_0 \int \mathbf{e}^\nu(x, b) \left[ -g(x) \right]^{1/2} \mathbf{d} b^i \]

This looks like we're adding vectors at different pts in space-time—usually meaningless: but \( \delta y \rightarrow \mathbf{j}^c(x) \) is indeed a vector since contributions just add at pt. B. More formally correct (using II propagators):
\[ \mathbf{j}^{\nu}(x) = \sum_{\delta} \epsilon_0 \int \mathbf{e}^\nu(x, b) \left[ -g(x, b) \right]^{1/2} \mathbf{g} \cdot X_{i8} \mathbf{d} b^i \quad (20) \]

Using (17) & (18) in the variational principle, (7) \( \Rightarrow \) Maxwell's Eq's:
\[ \mathbf{F}^{\nu\lambda} = 4\pi \mathbf{j} \]
(21)

The source-free Maxwell's eqn not included in (21) come from the defn, (14):
\[ \mathbf{F}^{\nu\lambda} = 0 \]
(22)

In summary, \( \frac{\partial \mathbf{F}}{\partial x^i} = 0 \) \( \Rightarrow \) eqns of motion, (23a)
\[ \omega = \text{NR. equivalent: } m \frac{\partial \mathbf{v}}{\partial t} = e \mathbf{E} + e \mathbf{v} \times \mathbf{B} \]
While \( \frac{\delta L}{\delta A_k} = 0 \Rightarrow \nabla \cdot \mathbf{E} = 4\pi \rho \) and \( \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} \)
(23b)
\[ \nabla \times \mathbf{E} = 0 \text{ and } \nabla \cdot \mathbf{E} + (\varepsilon_0 \varepsilon_r) \frac{\partial \mathbf{B}}{\partial t} = 0 \]
(23c)

**General Version of Energy-Momentum Tensors**

1) Write Scalar Lagrangian density, \( L \)
2) Vary the action w/ all entities introduced (as world-lines & potentials, above)
3) For the variation of each entity we get a corresponding "equation of motion"
But if an additional entity set in (12) which we’ve neglected so far: $g_{ik}$ appear in 1st 2 terms!

Try to evaluate $\frac{\delta A}{\delta g_{ik}}$: formally we have:

$$\delta A = -\frac{1}{2} \int \frac{\partial}{\partial g_{ik}} \delta g_{ik} \sqrt{-g} \, d^4 x$$  \hspace{1cm} (24)

This DEFINES the energy-momentum tensor for the entire physical system described by the action principles.

For our treated case: $T_{ik}^{\text{cm}} + T_{ik}^{\text{em}}$  \hspace{1cm} (25)

Take the variation, $\delta (g_{ik} g^{jk}) = \delta (\delta_{ij}) \Rightarrow$

$$\delta g_{ik} = -g_{ik} g_{jm} \delta g_{jm} \quad \text{so we write (24) as}$$

$$\delta A = \frac{1}{2} \int \frac{\partial}{\partial g_{ik}} \delta g_{ik} \sqrt{-g} \, d^4 x$$  \hspace{1cm} (26)

We’ll study $T_{ik}^{\text{cm}}$ and $T_{ik}^{\text{em}}$ in detail soon. First, get a formal expression for the general action described by $\mathcal{L}$.

Assume $\mathcal{L}$ contains $g_{ik}$ & $\delta g_{ik}$ through $\Gamma$’s needed to make the expression generally covariant.

Consider $g_{ik} \rightarrow g_{ik} + \delta g_{ik}$: we know $dg = g_{ki} \, dg^{ki}$

$$\delta \sqrt{-g} = \frac{1}{2} \sqrt{-g} g_{ik} \delta g_{ik}$$  \hspace{1cm} (27)

so then $\delta (L - E) = \int \frac{\partial}{\partial g_{ik}} \delta g_{ik} + \frac{\partial}{\partial \delta g_{ik}} \delta (g_{ik}) \sqrt{-g} \, d^4 x + \frac{1}{2} \sqrt{-g} g_{ik} \delta g_{ik}$

Since $\delta (g_{ik}) = (\delta g_{ik}) \, d$, we turn the middle piece into:

$$\int \frac{\partial}{\partial g_{ik}} \delta (g_{ik}) \, \sqrt{-g} \, d^4 x = \int \frac{\partial}{\partial g_{ik}} \sqrt{-g} g_{ik} d^4 x - \int \frac{\partial}{\partial g_{ik}} \sqrt{-g} \, d^4 x$$  \hspace{1cm} (28)

As usual for variational problems $\delta g_{ik} = 0$ on boundary of $\mathcal{V}$, so 1st term on RHS of (28) vanishes, and then

$$\delta A = \int \frac{\partial}{\partial g_{ik}} \left( \frac{\sqrt{-g}}{2} \left( g_{ik} \right) \right) \, d^4 x$$

Compare with (24) to see:

$$T_{ik} = 2 \left[ \frac{\sqrt{-g}}{2} \left( g_{ik} \right) \right]$$
Use (27) again to get \((\delta g)_{ik} = \frac{1}{2} \sqrt{g} \delta g_{ik} \) so that

\[
T^k_{ik} = 2 \left[ \left( \frac{\partial}{\partial g_{ik}} \right)_k + \frac{1}{2} \frac{\partial}{\partial g_{ik}} g^m_{jr} \left( g^l_{jm} \delta_{kl} - \frac{\partial}{\partial g_{kl}} - \frac{1}{2} g_{kl} \right) \right]
\]  

(27)

If \( \delta \) doesn't depend explicitly on \( g_{ik} \), then

\[
T^k_{ik} = -2 \frac{\partial}{\partial g_{ik}} \delta g_{ik}
\]  

(31)

Obviously, the same thing could be done for \( \delta g_{ik} \Rightarrow \)

\[
T^k_{ik} = 2 \frac{\partial}{\partial g_{ik}} \delta g_{ik}
\]  

(32)

Note \( T^k_{ik} = T^{(ik)} \) - automatically symmetric.

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**Energy-Momentum Tensors Useful in GR**

**(c)=1**

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### A: EM Field: Use 2nd term of (12) & (32) ...

\[
L = -\frac{1}{16\pi} F_{ik} F^{ik} = -\frac{1}{16\pi} F_{ik} F^{ik} \quad \text{km}
\]

so

\[
\frac{\partial}{\partial g_{ik}} = -\frac{1}{16\pi} F_{ij} F^{ij} = -\frac{1}{16\pi} F_{ik} F^{ik} \quad \text{& then}
\]

\[
T^k_{ik} = -\frac{1}{16\pi} \left[ F_{ik} F^{ik} - \frac{1}{4} F_{ij} F^{ij} \right]
\]  

(33)

Since the trace of the tensor, \( T = g_{ik} T^{ik} = T^i_i \) (34)

it's clear the trace of the EM E-M tensor vanishes.

Then, in SR case we find:

\[
F_{ik} = \begin{pmatrix}
O & -H_x & -E_x \\
-H_x & O & -E_y \\
-H_y & -E_x & O
\end{pmatrix}
\]  

(35)

and, then

\[
T^k_{ik} = \frac{1}{16\pi} \begin{pmatrix}
-2(E_x^2 + H_x^2) + E_x H_x & -E_x H_y & E_x E_y + H_x H_y & E_x H_y - H_x E_y \\
E_x H_y + H_x H_y & -2(E_y^2 + H_y^2) + E_y H_y & E_y E_z + H_y H_z & E_y H_z - H_y E_z \\
E_y H_z + H_y H_z & E_y E_z + H_y H_z & -2(E_z^2 + H_z^2) + E_z H_z & E_z H_z - H_z E_z \\
H_x E_z - H_z E_x & H_y E_x - H_z E_y & H_x E_y - H_z E_x & -E_x^2 + H_x^2
\end{pmatrix}
\]  

(36)
(b) **Radiation Field:** special case of (56) when \( |E| = |H| \) \( \quad \overline{E \cdot H} = 0 \) \hspace{1cm} (37)

If the radn is a plane wave in \( \mathbb{R} \) direction, only \( E_x \neq 0 \), \( H_y \neq 0 \) & \( E_x = H_y \)

\[
T^{(c)}_{(EM)} = \frac{1}{8\pi} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & E^2 + H^2 & -H_x E_y \\
0 & -H_y E_x & 0 & -E^2 - H^2 \\
0 & 0 & E_x H_y & -E^2 - H^2
\end{bmatrix}
\] \hspace{1cm} (38)

If it is black-body radn, it is isotropic & we have a simpler form: average (38) over all directions:

\[
T^{(c)}_{(EM)} = \text{diag} \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -u \right) \quad \text{where} \quad u = \frac{E^2 + H^2}{8\pi} \hspace{1cm} (39)
\]

(c) **System of Particles:** use 1st term of (12)

\[
\sum_a \overline{m_a d\alpha} = -\sum_a \int s_a \delta_q(x, A) d\alpha \, d^3x 
\] \hspace{1cm} (40)

Use

\[
\delta(\alpha^2) = \delta g_{ik} d\alpha^i d\alpha^k 
\] \hspace{1cm} (41)

\[
-\sum_a \int s_a \delta_a(x, A) \frac{i}{2} \frac{d\alpha^i}{\alpha^i} d^3x = \int \overline{d\alpha} \delta g_{ik} d\alpha^i d\alpha^k 
\] \hspace{1cm} (42)

From the definition (24) we get:

\[
T^{(c)}_{(m)} = \sum_a \int s_a \delta_q(x, A) \left[ -\overline{g}(x, A) \right]^{-\frac{1}{2}} \frac{i}{\sqrt{2}} \frac{d\alpha^i}{\alpha^i} d\alpha^k d^3x 
\] \hspace{1cm} (43)

We need these \( ii \) propagators to preserve covariance:

The trace of this tensor is:

\[
T^{(c)}_{(m)} = \sum_a s_a \delta_q(x, A) \left[ -\overline{g}(x, A) \right]^{-\frac{1}{2}} \quad \text{da} 
\] \hspace{1cm} (44)
D) **Dust**: stream of particles with no pressure, i.e., no random relative motion. Use the LICO at $\alpha$ is @ rest, we get from (43) in the neighborhood of this particle, $\delta a = dt$, so:

$$T^\alpha{}_{(m)} = \sum_a a_m \delta\rho(x, \alpha) dt = \rho_0$$  \hspace{1cm} (45)

All other components are 0. In a general frame, where $v^i$ has velocity $v^i$, we have:

$$T^i{}_k = \rho_0 v^i v^k$$  \hspace{1cm} (46)

where $\rho_0 = \frac{T}{(m)}$ is a scalar, the rest mass density of dust.

E) **Fluid**: stream of particles $\exists$ in any $\delta V$ 3 a velocity vector = ave. of fluid element. But $\exists$ random velocities too which yield a pressure. Again, choose LICO at $v_i = (0, 0, 0, 1)$ is a fluid element @ rest @ pt. $X$. Near $X$, let $v_0$ be fixed. 3-vel:

$$\frac{da^i}{da} = \left( \frac{\tilde{v}_0}{\sqrt{1 - \tilde{v}_0^2}}, \frac{-1}{\sqrt{1 - \tilde{v}_0^2}} \right)$$  \hspace{1cm} (47)

Suppose $|\tilde{v}_0| < 1$; writing $\tilde{v}_0 = [v_{x0}, v_{y0}, v_{z0}]$,

$$T^i = \sum_a a_m \delta\rho(x, \alpha) \tilde{v}_0^2 da = \sum_a \sum_m \delta\rho(x, \alpha) \tilde{v}_0^2 da$$  \hspace{1cm} (48)

But (45) says coef. is $\rho_0$, so:

$$T^i = \sum a_m \delta\rho(x, \alpha) \tilde{v}_0^2 da = \sum \delta\rho(x, \alpha) (1 + \tilde{v}_0^2) da$$  \hspace{1cm} (49)

$$\rho = \rho_0 + 3p = \rho \ldots \text{the matter density}$$
Thus, in the general (non-null, non-locally inertial frame):

\[ T^{ik} = (p + p) v^i v^k - p \delta^{ik} \]  
\[ (50) \]

and \( T^i = p - 3p = p_0 \)  
\[ (51) \]

### Conservation Laws

Since \( \phi \) is a scalar we get a key result: General:

\[ x^i \rightarrow x'^i = x^i - \xi^i \]  
\[ (52) \]

But, still:

\[ \delta \phi = 0 \]  
\[ (53) \]

However, this nullity arises from 2 changes. As C.S.

\[ \phi / \xi \] change as vectors, tensors, etc. change. But, if \( \phi / \xi \) satisfy field eqns from \( \delta \phi / \delta \phi = 0 \), then \( \delta \phi / \delta \phi \) changes in \( \xi \). Second change is in \( g / \xi \). How do they change?

\[ g_{ik}(x^j) = g_{ik}(x^j) \frac{\partial x^l}{\partial x^{i'}} \frac{\partial x^{i'}}{\partial x^k} = g_{ik}(x^j) \delta^{l}_{i'} \delta^l_k \delta^{i'}_{k'} \]

To 1st order in \( \delta x \) 2 derivs:

\[ g_{ik}(x^j) = g_{ik}(x^j) - \delta_{ik} \delta_{j}^l - \delta_{ik} \delta_{j}^l \]

Use this special change, \( \delta x = 0 \) & (24) to see:

\[ 0 = \int \frac{1}{2} T^{ik} (\dot{\xi}^i + \dot{\xi}^i) \sqrt{g} d\Sigma = \int \frac{1}{2} T^{ik} \dot{\xi}^i \sqrt{g} d\Sigma \]

As usual, assume \( \dot{\xi}^i = 0 \) on body \( \Sigma \) of \( U \) so:

\[ 0 = \int \frac{1}{2} T^{ik} \dot{\xi}^i \sqrt{g} d\Sigma = \int \frac{1}{2} T^{ik} \dot{\xi}^i \sqrt{g} d\Sigma - \int \frac{1}{2} T^{ik} \dot{\xi}^i \sqrt{g} d\Sigma \]

i.e.,

\[ \int \frac{1}{2} T^{ik} \dot{\xi}^i \sqrt{g} d\Sigma = 0 \]

Example: Dust, (55) \( \Rightarrow \)

\[ 0 = (p v^i v^i) \nu_k = (p v^i) v^i + p v^i v^k \nu_k \]

Mult by \( v^i \) & use results \( v^i v^i = 1 \), \( v^i \nu_k = 0 \) \( \Rightarrow \)

\[ (55) \]

\[ (56) \]

Use (55) in (55) to see:

\[ v^i \nu_i = 0 \]  
\[ (58) \]

i.e., dust follows geodesics
Conformal Invariance

So far we've demanded theories should be invariant under general covariant trans. if another one, w/ no simple analog (Lorentz) in flat 5-T.

A region of S-T has: \[ ds^2 = g_{ik}dx^i dx^k \] (58)

At pt. P(x\(^\mu\)) use (58) to compare ratios of infinitesimal lengths or time-intervals. How to compare @ Q? Could basic units change, perhaps in poorly determined way? To allow for this, introduce arbitrary, 2x diff. func. \( \Omega(x) \) <

\[ d\bar{s}^2 = \Omega^2(\Omega) ds^2 = \Omega^2 g_{ik} dx^i dx^k \] (60)

Use \( d\bar{s} \)s for length meas, abitribariness hidden in \( \Omega \); for a theory to work w/ \( \Omega \) floating around, the eqns can't contain refs. to \( \Omega \), i.e. \( \Omega \rightarrow \bar{\Omega} \) should leave the theory unchanged. Conformal invariance.

A conformal transformation: \( \bar{g}_{ik} = \Omega^2 g_{ik} \)

Example: Maxwell's ELM is C.I. - look @ 2nd & 3rd parts of action in (12) & suppose under (61) \( A^i \) is unchanged:

\( A^i = \bar{A}^i \Rightarrow F^i = \bar{F}^i \) as coords don't change under C.T.

\( \bar{F}^i = g_{ik} \bar{F}^k \), \( F^i = \Omega^{-1} F^k \) but \( d\bar{s}^2 = \Omega^2 \sqrt{-g} \) so both tensors saa

Conformal Curvature Tensor: Let \( \Omega = \exp\phi \), \( \Delta \bar{s} = \phi_{ij} \phi_{ik} \)

\[ \Delta_{ik} = \bar{F}^i \bar{F}^j - \frac{1}{2} \delta_{ij} \bar{F}_{ik} \quad \Delta S = \phi_{ik} \phi_{jk} \quad \text{Then:} \]

\[ \bar{R}_{iklm} = \exp(2\phi) \left[ R_{iklm} + g_{im}g_{ln} \phi_{kl} + g_{im}g_{jl} \phi_{kl} + g_{il}g_{jm} \phi_{km} + g_{il}g_{kj} \phi_{jm} + g_{jm}g_{il} \phi_{mk} + g_{jl}g_{im} \phi_{mk} \right] \]

\[ R_{iklm} = \bar{R}_{iklm} + 2 \phi_{im} \phi_{ln} + 2 \delta_{ik} \phi_{lm} \] & \( R = \exp(2\phi) \left[ R + 6 \Delta S + 6 \Delta S \right] \)

None of these is C.I., but a combo, Weyl conf. curvature tensor, is:

\[ C^i_{\ jk} = R^i_{\ jk} + \frac{1}{2} \left[ g^i_k R_{ij} - g^i_j R_{ik} + g_{ij} R^k_{\ kl} - g_{ij} R^k_{\ l} - g_{ij} R^k_{\ l} + \frac{1}{2} R (g_{ij} g_{kl} - g_{ik} g_{lj}) \right] \]

Only C.T. leaving space flat is trivial, \( S = \text{const} \). But \( C^i_{\ jk} = 0 \) is conformally flat S-T.