This is an open book, take-home examination, covering many topics, but stressing the most recent. You may want to locate sources other than your texts and notes, so get started on it promptly. It must be turned in to me by 10:00 AM on Thursday, December 9th, but I’d be very happy if you hand it in during the last class on December 8th. I need to turn in the grades by the end of Friday, December 10th.

Collaboration between students (current and current or current and former) is not allowed. If you have a question, please address it to me. My e-mail is wiita@chara.gsu.edu, my office number is (404) 651-1367 and my home number is (609) 273-7177.

YOUR EXAM WILL NOT BE ACCEPTED UNLESS YOU SIGN THE FOLLOWING STATEMENT: I HAVE NEITHER GIVEN NOR RECEIVED ANY ASSISTANCE IN THIS EXAMINATION:

Please pay attention to these instructions:

1) If you use a fact, equation, etc., that is not from the text or the notes, please be sure to cite the book or journal title, author(s), and (volume and) page numbers, from which you obtained that information. Citing equations from the text or notes would also be appreciated.

2) Start each full question on a separate sheet and put your name on each sheet handed in, even though you will have stapled them together in the correct order before you give them to me. Clearly state any assumptions you make, and clearly label any diagrams you draw.

3) The point value of each question is given in brackets after the question or part thereof; they total 150. Partial credit will be awarded, so do as much as you can on each question. However, do not merely summarize scratch results. A correct answer is thoroughly presented, with every step carefully expounded. Also note that legibility is important; if I can’t read it, it can’t be right!

1. Special Relativistic Kinematics

What is the smallest kinetic energy, \( T_{e,\text{con}} \), that an electron must have in order to be able to impart half of this kinetic energy to a proton — originally at rest — in an elastic head on collision?

a) Find a single equation for the above situation that can be solved to determine the single, dimensionless, unknown quantity \( T_e/m_p \). [8]

b) Find the value of \( T_{e,\text{con}} \) in MeV using the approximate value \( m_p c^2 = 938 \text{ MeV} \). If you solve this equation by approximation, estimate the error in your answer. [5]
2. Tests of General Relativity

a) Compute the general relativistic time delay involved in bouncing a radar beam sent from Earth off of Mars at conjunction. Assume both planets are at their mean distances from the Sun. What fraction of the entire round trip time does this delay comprise? [5]

b) Describe the concept of gravitational lensing. Derive a formula for the angular bending induced by a point lens of mass $M$ at a distance $D_{OL}$ from the observer and a distance $D_{LS}$ from the source, which is at the distance $D_{OS}$ from the observer. By what factor might an object’s brightness be enhanced by having a massive object intersect the line of sight between the observer and the source? [10]

3. Simple Quantities of Fundamental Importance

Construct a length, a mass, a time and a density, purely by dimensional analysis, from the fundamental physical constants $c, G$ and $\hbar$. Express the values of these “Planck” lengths, masses, times, and densities in cgs units. [9]

4. Stress-Energy Tensor

An infinestimally thin rod of length $2a$ has a point mass $m$ at each of its ends. The center of the rod is fixed in the laboratory and the rod rotates about this point with an angular velocity $\Omega$ which is relativistic (i.e., $\Omega a \approx c$). Assume the rod is massless. What is $T^{ik}$ for the rod and particle system? [16]

5. Motion in a Schwarzschild Metric

Consider a non-rotating neutron star (or black hole). Derive the equations of motion relating $t, r$ and $\tau$ for a particle falling radially in the Schwarzshild geometry if it is released from rest at $r = R$. [15]

6. Black Holes

a) Show that a rocket ship which has once crossed the gravitational radius (event horizon) of a Schwarzschild black hole, must reach $r = 0$ in a proper time $\tau \leq \pi M$, regardless of how its engines are fired. [10]

b) Use Hawking’s area theorem to find the minimum mass $M_2$ of a Schwarzschild black hole that results from the collision of two Kerr black holes of equal mass $M$ and opposite angular momentum parameter $a$. Show that if $a \rightarrow M$, 50% of the rest mass could be radiated away. Show that if $a = 0$ the maximum efficiency is 29%. [Note, however, that numerical calculations for this case give an actual efficiency of $\sim 0.1\%$.] [10]
7. Fundamental Cosmological Equations

Start from (i) the Cosmological Principle, which leads to the conformally flat version of the Robertson-Walker line element,

\[ ds^2 = dt^2 - R(t)^2 \left[ dr^2 + r^2 d\Omega^2 \right] \left(1 + kr^2/4\right)^2, \]

(ii) the assumption that the universe is a perfect fluid, so

\[ T_{ik} = (\rho + p)u_i u_k - pg_{ik}, \]

and (iii) General Relativity with the cosmological term

\[ G_{ik} - \Lambda g_{ik} = 8\pi T_{ik}. \]

Then, use the fact that one can choose a local inertial coordinate system where \( u^i = (1, 0, 0, 0) \), to show that the field equations lead to two independent equations: [17]

\[ 3 \frac{\dot{R}^2 + k}{R^2} - \Lambda = 8\pi \rho(t), \]

\[ \frac{2R\ddot{R} + \dot{R}^2 + k}{R^2} - \Lambda = -8\pi p(t), \]

where we have taken \( G = c = 1 \) and a dot denotes differentiation w.r.t. \( t \). Obviously, \( \rho \) and \( p \) are functions only of \( t \) from our assumptions of homogeneity and isotropy in the Cosmological Principle.

8. Relativistic Cosmologies

a) A universe is spatially flat and contains both matter and a cosmological constant. For what value of \( \Omega_{m,0} \) (the current value of \( \Omega_m \)) is the age of the universe, \( T_0 \equiv H_0^{-1} \)? [5]

Now consider a consensus model cosmology with \( \Omega_{m,0} = 0.30, \Omega_{\Lambda,0} \approx 0.70, \Omega_{b,0} = 0.04 \) and \( \Omega_{\gamma,0} = 5.0 \times 10^{-5} \).

b) When was \( \Omega_m = \Omega_\Lambda \)? Express this in terms of the value of the scale factor \( a \equiv R/R_0 \) and in units of Gyr. [5]

c) How old is this universe? [3]

d) What is the total mass of all matter within our horizon? What is the total energy of all photons within our horizon? How many baryons are within our horizon? [6]
9. Dark Matter

a) Suppose it were suggested that mini-black holes of mass $10^{-8}M_\odot$ made up the dark matter in the halo of our galaxy. How far away would the nearest such BH-ette be, and how often would one pass within 1 AU of the sun? (Order of magnitude estimates would be fine here.) [6]

b) The Draco galaxy is a dwarf galaxy within the Local Group. It has a luminosity, $L_D = 1.8 \pm 0.8 \times 10^5 L_\odot$. One half of its total mass is within a radius $r_h = 120 \pm 12$ pc. The red giants in Draco are bright enough to have their line-of-sight velocities measured, and the radial dispersion for them is $\sigma_r = 10.5 \pm 2.2$ km s$^{-1}$. What are the mass and M/L for Draco? (Express the latter in units of $M_\odot/L_\odot$.) [10]

10. Growth of Perturbations in the Universe

Consider a critical-density universe in which massive neutrinos contribute $\Omega_\nu$ to the density parameter. Show that, on scales smaller than the neutrino Jeans length, that perturbations in the remaining cold component grow as $\delta \propto t^\alpha$, where [10]

$$\alpha = \frac{(25 - 24\Omega_\nu)^{1/2} - 1}{6}.$$