Viscosity... it's such a drag!

- Viscosity transports momentum between parts of the fluid that are in relative motion
- Viscosity results from particle collisions
- The momentum equation with viscosity is given by

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \frac{\mathbf{J} \times \mathbf{B}}{c} - \nabla p + \rho \nu \nabla^2 \mathbf{V} \tag{13}$$

Here, ν is the *kinematic viscosity*

Putting the momentum equation in dimensionless form

- Define $\mathbf{V} \equiv V_0 \tilde{\mathbf{V}}$... where V_0 is a characteristic value and $\tilde{\mathbf{V}}$ is dimensionless, and use $V_0 \equiv L_0/t_0$
- Use these definitions in the momentum equation while including only the viscosity term on the RHS

$$\rho \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = \rho \nu \nabla^2 \mathbf{V} \tag{14}$$

$$\frac{V_0}{t_0} \frac{\partial \tilde{\mathbf{V}}}{\partial \tilde{t}} + \frac{V_0^2}{L_0} \tilde{\mathbf{V}} \cdot \tilde{\nabla} \tilde{\mathbf{V}} = \frac{\nu V_0}{L_0^2} \tilde{\nabla}^2 \tilde{\mathbf{V}}$$

$$\frac{\partial \tilde{\mathbf{V}}}{\partial \tilde{t}} + \tilde{\mathbf{V}} \cdot \tilde{\nabla} \tilde{\mathbf{V}} = \frac{\nu}{V_0 L_0} \tilde{\nabla}^2 \tilde{\mathbf{V}} \tag{15}$$

where the Reynolds number is

$$\mathrm{Re} \equiv \frac{V_0 L_0}{\nu} \tag{16}$$

The Reynolds number gauges the importance of the advective term compared to the viscous term

- We can rewrite the momentum equation with only viscosity as

$$\frac{\partial \tilde{\mathbf{V}}}{\partial \tilde{t}} + \underbrace{\tilde{\mathbf{V}} \cdot \tilde{\nabla} \tilde{\mathbf{V}}}_{\text{advection}} = \underbrace{\frac{1}{\mathrm{Re}} \tilde{\nabla}^2 \tilde{\mathbf{V}}}_{\text{viscous diffusion}} \tag{17}$$

- A necessary condition for turbulence to occur is if

$$\mathrm{Re} \gg 1 \tag{18}$$

  so that the viscous term is negligible on scales $\sim L_0$.
  - In astrophysics, usually $\mathrm{Re} \equiv \frac{L_0 V_0}{\nu} \ggg\gg 1$
- The only scale in this equation is the viscous scale

Physics of cooking, part 2!

- The Reynolds number is

$$\mathrm{Re} \equiv \frac{L_0 V_0}{\nu} \tag{19}$$

- If you want to make peanut butter turbulent, you can either
  - Get a humongous vat of it (increase L_0), or
  - Stir it up really quickly (increase V_0)
- Alternatively, since viscosity is a function of temperature (as a in a plasma), you could also heat it up (decrease ν)

Putting the induction equation in dimensionless form

- The induction equation with uniform resistivity is

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + D_\eta \nabla^2 \mathbf{B} \tag{20}$$

- Again, define $\mathbf{B} \equiv B_0 \tilde{\mathbf{B}} \ldots$ with $V_0 \equiv L_0 / t_0$

$$\frac{B_0}{t_0} \frac{\partial \tilde{B}}{\partial \tilde{t}} = \frac{V_0 B_0}{L_0} \tilde{\nabla} \times \left(\tilde{\mathbf{V}} \times \tilde{\mathbf{B}} \right) + \frac{D_\eta B_0}{L_0^2} \tilde{\nabla}^2 \tilde{\mathbf{B}} \tag{21}$$

$$\frac{\partial \tilde{B}}{\partial \tilde{t}} = \tilde{\nabla} \times \left(\tilde{\mathbf{V}} \times \tilde{\mathbf{B}} \right) + \frac{D_\eta}{L_0 V_0} \tilde{\nabla}^2 \tilde{\mathbf{B}} \tag{22}$$

Defining the magnetic Reynolds number and Lundquist number

- We define the *magnetic Reynolds number* as

$$\mathrm{Rm} \equiv \frac{L_0 V_0}{D_\eta} \tag{23}$$

- The Alfvén speed is

$$V_A \equiv \frac{B}{\sqrt{4\pi\rho}} \tag{24}$$

- We define the *Lundquist number* as

$$S \equiv \frac{L_0 V_A}{D_\eta} \tag{25}$$

  where we use that the characteristic speed for MHD is $V_0 = V_A$

Rm and S gauge the relative importance between the advection term and the resistive diffusion term

- We can write the induction equation as

$$\frac{\partial \tilde{B}}{\partial \tilde{t}} = \tilde{\nabla} \times \left(\tilde{\mathbf{V}} \times \tilde{\mathbf{B}} \right) + \frac{1}{\mathrm{Rm}} \tilde{\nabla}^2 \tilde{\mathbf{B}} \tag{26}$$

  - If $\mathrm{Rm} \gg 1$ then advection is more important (ideal MHD limit)
  - IF $\mathrm{Rm} \ll 1$ then diffusion is more important
- Usually in astrophysics, $\mathrm{Rm} \ggg\gg 1$
  - Example: $T \sim 10^4$ K, $L_0 \sim 1$ pc, $V \sim 1$ km/s. Then

$$\mathrm{Rm} \sim 10^{16}$$

Interstellar plasmas are extremely highly conducting!

Re, Rm, and S can also be expressed as the ratio of timescales

- The Reynolds number is

$$\mathrm{Re} \equiv \frac{L_0 V_0}{\nu} = \frac{\text{viscous timescale}}{\text{advection timescale}} \tag{27}$$

- The magnetic Reynolds number is

$$\mathrm{Rm} \equiv \frac{L_0 V_0}{D_\eta} = \frac{\text{resistive diffusion timescale}}{\text{advection timescale}} \tag{28}$$

- The Lundquist number is

$$S \equiv \frac{L_0 V_A}{D_\eta} = \frac{\text{resistive diffusion timescale}}{\text{Alfvén wave crossing time}} \tag{29}$$

The relative importance of viscosity vs. resistivity is given by the magnetic Prandtl number

- The magnetic Prandtl number is

$$\mathrm{Pm} \equiv \frac{\nu}{D_\eta} = \frac{\text{resistive diffusion timescale}}{\text{viscous timescale}} \tag{30}$$

  where ν and D_η are both in units of a diffusivity: length^2/time
- Usually $\mathrm{Pm} \not\approx 1$, but people doing simulations (like me) often set $\mathrm{Pm} = 1$ for simplicity (sigh)
- In plasma turbulence, the Prandtl number determines which scale is larger: the viscous dissipation scale or the resistive dissipation scale
  - Helps determine the physics behind dissipation of energy in turbulence!

Is there flux freezing in resistive MHD?

- Short answer: no!
- Long answer: let's modify the frozen-flux derivation!
- The change of flux through a co-moving surface bounded by a contour $\mathcal{C}$ is

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = -c \oint_{\mathcal{C}} \left(\mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} \right) \cdot \mathrm{d}\boldsymbol{l} \tag{31}$$

- In ideal MHD, the integrand is identically zero.
- In resistive MHD, the integrand is $\eta \mathbf{J}$!
- The change in flux becomes

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = -c \oint_{\mathcal{C}} \eta \mathbf{J} \cdot \mathrm{d}\boldsymbol{l} \tag{32}$$

This is generally $\neq 0$, so flux is not frozen-in.

Viscous and resistive heating

- Ohmic (resistive) heating is given by

$$Q_\eta = \mathbf{E} \cdot \mathbf{J} = \eta J^2 \tag{33}$$

  This shows up as a source term in the energy equation
- Heating preferentially occurs in regions of very strong current, but those regions typically have small volumes
- Viscous heating is of the form

$$Q_\nu = \rho\nu\nabla\mathbf{V}^T : \nabla\mathbf{V} \tag{34}$$

Key Properties of Resistive MHD

- Magnetic topology is not preserved
- Mass, momentum, and energy are conserved
- Helicity and cross-helicity are approximately conserved
- There is Ohmic (resistive) and viscous heating
- There's dissipation!
- The scales in the problem are set by viscosity and resistivity