5.2 Navier–Stokes equation

On substituting (5.1) in (4.8) with π_{ij} given by (5.7), we obtain

$$\rho \frac{dv_i}{dt} = \rho F_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{v} \right) \right]. \tag{5.8}$$

This equation is known traditionally as the *Navier–Stokes equation*, although several persons in the early nineteenth century made contributions towards establishing this equation (Navier 1822; Poisson 1829; Saint-Venant 1843; Stokes 1845).

We shall mostly be concerned with situations in which the spatial variation of u is not important. Hence u can be taken outside the spatial derivative so that

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{F} - \nabla p + \mu \left[\nabla^2 \mathbf{v} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{v}) \right]. \tag{5.9}$$

The term involving $\nabla(\nabla \cdot \mathbf{v})$ has significance only in the case of flows with variable compression. For example, when we are studying the viscous dissipation of acoustic waves, this term has to be taken into account. For most fluid flow problems of interest to us, however, it can neglected. Hence, by Navier–Stokes equation, we shall usually mean the simpler version of the equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{F} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v}, \tag{5.10}$$

where

$$\nu = \mu / \rho \tag{5.11}$$

is called the *kinematic viscosity*. We note that (5.10) is the same as (3.52).

The version (5.10) of the Navier–Stokes equation differs from the Euler equation (4.11) only by virtue of the additional term $\nu \nabla^2 \mathbf{v}$. The mathematical characters of the two equations, however, are vastly different due to the fact that the Navier–Stokes contains a higher spatial derivative than the Euler equation. Hence the Navier–Stokes equation requires more boundary conditions for solutions in finite regions. While solving ideal fluid problems with solid boundaries, one usually takes the normal component of velocity to be zero on the solid surface. We have seen in §4.7 that this suffices to find the solution uniquely and additional boundary conditions cannot be implemented. On the other hand, while solving viscous fluid problems, one imposes the extra boundary condition on a solid surface that the tangential component of velocity is also zero making $\mathbf{v} = 0$ there. Observations from everyday life convince us that this is actually the