PLASMA 8120 3/18/2021

DERIVATION OF THE 2.5 D DYNAMO MODEL

(1) BASIC ASSUMPTIONS:
A) KINEMATIC, i.e. FLOWS INSIDE THE SUN DRIVE THE DYNAMO .. THE BACK-REACTION OF THE MAGNETIC FIELD ON THE FLOW (VIA $\vec{J}\times\vec{B}$) is IGNORED. JUSTIFICATION; DYNAMIC PLASMA $\beta = \frac{1}{2}\rho V^2/(B^2/8\pi) >> 1$.
B) TOROIDAL SYMMETRY, i.e. $\partial_\varphi = 0$

(2) We work in spherical geometry, $\hat{\theta}$ in the N-S DIRECTION (poloidal), $\hat{\varphi}$ in the E-W DIRECTION (TOROIDAL)

(3) Eqs. TO SOLVE:

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{V}\times\vec{B}) - \eta \vec{J}$$

$$\vec{J} = \vec{\nabla}\times\vec{B} \quad , \quad \vec{\nabla}\cdot\vec{B} = 0$$

$\vec{V}(r,\theta)$ is PRESCRIBED, E.g. DETERMINED FROM HELIOSEISMOLOGY
$\vec{V}_{poloidal}$ = MERIDIONAL FLOW (SMALL)
$V_\varphi$ = DIFFERENTIAL ROTATION (LARGE)

(4) MATHEMATICAL SIMPLIFICATION:
$\vec{\nabla}\cdot\vec{B} = 0$ is DIFFICULT TO MAINTAIN IN SIMULATION

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2.5 D Dynamo Eqs.

(4) Continued

So, use vector potential (in part), $\vec{B} = \vec{\nabla} \times \vec{A}$

$$(\vec{\nabla} \times \vec{A})_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\varphi) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\varphi} = 0$$

$$(\vec{\nabla} \times \vec{A})_\theta = \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi}_{=0} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi)$$

$(\vec{\nabla} \times \vec{A})_\varphi = B_\varphi$ ⇒ we won't use this

So $B_r$ and $B_\theta$ are expressed in a single component of the vector potential, $A_\varphi$

(5) How about $\vec{\nabla} \cdot \vec{B} = 0$, that should be satisfied now automatically

$$\vec{\nabla} \cdot \vec{B} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 B_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\theta) + \frac{1}{r\sin\theta}\frac{\partial B_\varphi}{\partial\varphi}_{=0}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{r}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\varphi)\right] - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{\sin\theta}{r}\frac{\partial}{\partial r}(rA_\varphi)\right]$$

$$= 0$$

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(5) Continued...
So, we have reduced the system from three unknowns ($B_r, B_\theta, B_\varphi$) to two unknown functions, ($A_\varphi, B_\varphi$) with $\vec{\nabla} \cdot \vec{B} = 0$ automatically satisfied. That greatly simplifies numerical solutions

(6) Basic Eqs:

See the slide, #11 in the March 3 lecture.
Note: a) The term with $\Omega$ in it in the 2nd equation represents the wrapping up of field lines through differential rotation
b) The dissipation terms, that have $\eta$ in them, represent the decay of the field
c) $S_\alpha$ is the $\alpha$-effect, a representation of an essentially 3D effect, driven by the Coriolis force and meridional circulation. It represents the re-generation of poloidal field from toroidal field.

$$S_\alpha = \alpha B_\varphi$$

$\alpha$ can be a function, i.e. $\alpha = \alpha(r, \theta)$
$\Rightarrow$ without $S_\alpha$ the dynamo peters out (Cowling's theorem)