Magnethydrodynamics and Plasma Physics
Notes: February 11, 2019
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FORCE FREE FIELDS (FFF)

- No magnetic force. $\bar{J}x\bar{B} = 0 \ \rightarrow \ only\ when\ Lorentz\ force\ dominates$
- Plasma β= $\frac{8\pi}{B^2}P \quad \rightarrow \quad where\ P\ is\ thermal\ energy\ and\ \frac{8\pi}{B^2}\ is\ magnetic\ energy$

If β << 1 Magnetic field dominates
If β >> 1 Thermal energy dominates (kinetic: $\frac{1}{2}pv^2 \gg \frac{B^2}{8\pi}$)

- When β << 1 (and fluid frozen-in: R_m >> 1) Plasma follows the field lines, such as in the solar corona
- The plasma drives the such as the interior of stars: Dynamo Action
- Dynamo Action: motion of plasma wraps up field lines (gives us the solar cycle)

STATIC FORM EQUATION:

$$\bar{J}x\bar{B} - \bar{\nabla}\ P = 0$$

$$\frac{B^2}{l_1} - \frac{P}{l_2} = 0$$

$$\boldsymbol{\beta = \frac{P}{B^2} * \frac{L}{l} \sim 1}$$

FORCE FREE FIELD LINES: (β << 1) ($\boldsymbol{\bar{J}x\bar{B} = 0}$)

Non-linear partial differential equation in the first order:

$$\hat{z} = j_x B_y - j_y B_x = 0$$
$$(\partial y B_z - \partial x B_y)B_y + (\partial x B_x - \partial x B_z)B_x = 0$$

$$\bar{\nabla} * \bar{B} = \partial x B_x + \partial y B_y + \partial z B_z 0$$

SPECIAL SOLUTIONS

$$\bar{J} x \bar{B} = 0 \quad \rightarrow \quad \bar{J} = \lambda(\bar{r})\bar{B}$$
$$\bar{\nabla} * \bar{J} = \bar{\nabla} * (\bar{\nabla} x \bar{B}) = 0 \quad \rightarrow \quad \lambda(\bar{r})\,\bar{\nabla} * \bar{B} + \bar{\nabla}\lambda * \bar{B}$$
$$\bar{\nabla}\lambda * \bar{B} = 0 \quad \rightarrow \quad \lambda \; constant \; along \; field \; line$$

Potential Fields: No Current (Curl)

$$(\bar{\nabla} x \bar{B}) = 0 \quad \rightarrow \quad \bar{B} = \bar{\nabla}\varphi \; (\bar{\nabla} \; x \; \bar{\nabla} \; \varphi \; = 0)$$
$$(\bar{\nabla} x \bar{B}) = 0 \quad \rightarrow \quad \bar{\nabla} * (\bar{\nabla}\varphi) \; = \; \Delta\varphi \; = \; 0$$
$$\varphi = harmonic \; function \; matching \; boundary \; conditions$$

Vector Potential

$$\bar{B} \equiv \bar{\nabla} x \bar{A} \quad \rightarrow \quad \bar{\nabla} * \bar{B} = \bar{\nabla} * (\bar{\nabla} x \bar{A}) = 0$$

2D Potential Fields: $\boldsymbol{B_z = 0}$; $\boldsymbol{\partial z = 0}$

$$B_x(x,y), B_y(x,y) \;\; ; \;\; B_x = \partial y A_z(x,y) \;\; ; \;\; B_y = -\partial x A_z(x,y)$$
$$A = vector \; potential\text{:} \;\; \bar{\nabla} x \bar{B} = 0 \quad \rightarrow \quad \Delta A_z = 0$$
$$Assume \; A_z = R_e f(x + iy) = R_e f(z)$$
$$if \; A_z = R_e f(z), \;\; then \;\; \Delta A_z = 0$$

Klebsch Coordinates

- Define: $\bar{B} = \bar{\nabla}\alpha \; x \; \bar{\nabla}\beta, \;\; \alpha(\bar{r}), \beta(\bar{r})$
- So now we have reduced the problem to two unknown functions instead of three (B_x, B_y, B_z)

$$\bar{\nabla}\alpha * \; \bar{B} = 0 \quad \rightarrow \quad \bar{\nabla}\beta * \bar{B} = 0.$$

- Hence α and β are constant along field lines

- There is no generating function for $\bar{J} x \bar{B} = 0$ but what about $\bar{J} * \bar{B} = 0$?
- Maximum force field: $|\bar{J} x \bar{B}| = |\bar{J}||\bar{B}|$

  Assume $\bar{B} = \alpha \bar{\nabla} \beta \quad \rightarrow \quad \bar{J} = \bar{\nabla} \alpha \; x \; \bar{\nabla} \beta$

Helicity in Klebsch Coordinates

$$H \cong \int_V \bar{A} * \bar{B} dV \quad ; \quad \bar{B} = \bar{\nabla} \; x \; \bar{A} ; \qquad \bar{\nabla} * \bar{B} = 0$$

$$\bar{A} = \frac{1}{2} (\alpha \bar{\nabla} \beta - \beta \bar{\nabla} \alpha) \quad \rightarrow \qquad \bar{B} = \bar{\nabla} \alpha \; x \; \bar{\nabla} \beta$$

- Helicity = 0 in Klebsch coordinates…what is helicity? Winding of field lines ex. Tokamak

Cylinder Symmetry: Force Field Fields

- Twisted Fields on Cylinder Shells: $\partial z = 0$, $\partial \varphi = 0$

$$\bar{\nabla} * \bar{B} = 0 = \frac{1}{r} \partial r (r B_r) = 0 \qquad B_r = 0$$

$$\bar{B} = \left(B_r, B_\varphi, B_z\right) = (0, B_\varphi, B_z)$$

$$\bar{J} x \bar{B} = 0 \quad : \quad Force\ Free\ Field$$

$$\frac{d}{dr}\left(B_\varphi^2 + B_x^2\right) + \frac{1}{r} B_\varphi^2 = 0 \qquad B_\varphi^2 + B_x^2 : magnetic\ pressure, \qquad \frac{1}{r} B_\varphi^2 : magnetic\ tension$$

- Take Arbitrary function: $B^2(r) = f(r)$

$$B_y^2 = -\frac{r}{2} \frac{\partial f}{\partial r} + f \qquad , \qquad B_z^2 = f(r) - B_y^2$$

- Example: uniform twist field $\quad B_y = \frac{B_0 (\Phi^{r}/_{2c})}{1 + \Phi^2 (r^2/_{(2c)^2})} \qquad B_x = B_y (^{2L}/_{\Phi r})$
- Here we have a general solution: Define a function f(r), with some mild conditions.

Linear FFF

- Recall: $\bar{J} = \alpha(\bar{r}) \bar{B}$

Assume $\alpha(\bar{r}) = constant$

$$\overline{\nabla}\, x\, \overline{B} = \alpha\overline{B}$$

$$\overline{\nabla}\, x\, (\overline{\nabla}\, x\, \overline{B}) = \alpha\overline{\nabla}\, x\, \overline{B} = \alpha^2\overline{B} = \Delta^2\overline{B}$$

$$(\overline{\nabla}^2 + \bar{\alpha}^2)\overline{B} = 0$$

- The Helmholtz equation is linear. It has general solutions in the form of separation of variables, including exponentials with complex exponents in cartesian coordinates

HOMEWORK

1) Calculate $\bar{J}$ in Klebsch: instead of $\bar{J}x\overline{B} = 0$ but what about $\bar{J} * \overline{B} = 0$?
2) Try out a few functions (quartic, quadratic, gaussian) to see how B_φ &. B_z work out