Helicity is NOT a local quantity

(1) You may have noticed, when talking about helicity, I have mostly used the expression $\int_V \vec{A}\cdot\vec{B}\,dV$, NOT $\vec{A}\cdot\vec{B}$

Why is that?

(2) Consider $\vec{B}' = \vec{\nabla}\times\vec{A}'$ & $\vec{B} = \vec{\nabla}\times\vec{A}$ with $\vec{A} = \vec{A}' + \vec{\nabla}\varphi$, then:

$$\vec{B} = \vec{B}' \quad \& \quad \vec{A} \neq \vec{A}' \quad (\vec{\nabla}\cdot\vec{B}' = \vec{\nabla}\cdot\vec{B} = 0)$$

So you have EXACTLY the same magnetic fields, but not the same helicity? How can that be?

(3) Consider $H' - H = \int_V \vec{A}'\cdot\vec{B}'\,dV - \int_V \vec{A}\cdot\vec{B}\,dV$
(same volume of course)

$$\Delta H = H' - H = -\int_V (\vec{\nabla}\varphi\cdot\vec{B})\,dV$$

Use, once more, a vector identity

$$\vec{\nabla}\cdot(\varphi\vec{B}) = \vec{\nabla}\varphi\cdot\vec{B} + \varphi\underbrace{\vec{\nabla}\cdot\vec{B}}_{=0} = \vec{\nabla}\varphi\cdot\vec{B}$$

Hence: $$\Delta H = \int_V \nabla\cdot(\varphi\vec{B})\,dV$$

Again, use Karl Friedrich Gauss' law
(1777–1855)

$$\int_V \vec{\nabla}\cdot(\varphi\vec{B})\,dv = \int_S (\varphi B)\cdot d\vec{o}$$

HENCE $\Delta H = \int (\varphi \vec{B})\cdot d\vec{o}$

SO, ONLY THE COMPONENT OF $\vec{B}$ PERPENDICULAR TO THE SURFACE CONTRIBUTES. IF THERE IS ONLY A PARALLEL COMPONENT OF $\vec{B}$ TO THE SURFACE $(\vec{B}\cdot d\vec{o}=0)$, $\Delta H = 0$, SO H IS WELL DEFINED. ELSE IT IS NOT IN A BOUNDED VOLUME.

(4) CONSEQUENCES:

- THE OUTER FLUX SURFACE (OR OUTSIDE OF INSTRUMENT TORUS) OF A TOKAMAK:
  $\vec{B}\cdot d\vec{o} = 0 \rightarrow H$ WELL DEFINED
- THE UNIVERSE: THE UNIVERSE HAS NO BOUNDARY, HENCE H IS WELL DEFINED (PFFFU...)
- STELLAR CORONAE, THE SUN AND OTHER STARS WITH MAGNETIC FIELDS IN THEIR PHOTOSPHERES, HAVE MAGNETIC FIELDS PIERCING THROUGH THEIR SURFACES, E.G. STARSPOTS. SO FOR STELLAR CORONAE $\vec{B}\cdot d\vec{o} \neq 0$ EVERYWHERE.
  SO WHAT NOW?
- AND WHAT ABOUT THE HELICITY OF THE ~~Mag Helic~~ HELICITY OF THE HELIOSPHERE AS A WHOLE?

(5) Helicity is NOT a local quantity

So, the Helicity of stellar coronae depends on φ on the boundary, the interface with the photosphere and that with the interstellar medium

But φ and $\vec{A}$ cannot be measured! The magnetic field $\vec{B}$, both its direction and magnitude can be measured, but $\vec{A}$ (or $\vec{A}'$) cannot. Hence Helicity cannot be determined if there is ~~a~~ (part of) the boundary with $\vec{B} \cdot d\vec{\sigma} \neq 0$

(6) Illustration.. Compare a ring and a Moebius ring (see illustrations). Let the magnetic field lines all be inside the ring or Moebius ring. Hence the Helicity of the structure can be determined, and it is different for the two structures

A) Look at the 3D picture: A Moebius ring has only one surface!

B) Now look at the Moebius wedding ring. Imagine the untwisted half of the ring sticking out above a stellar photosphere.

$\Rightarrow$ There is no way of telling what kind of ring

This is FROM THE CORONAL PORTION: REGULAR RING OR MOEBIUS RING! THE HELICITY IS UNDETERMINED $\Rightarrow$ HELICITY IS A NONLOCAL QUANTITY

(7) RELATION OF HELICITY TO LINKAGE

Linkage: AS AN EXAMPLE, TAKE THE ORBIT OF THE EARTH AND THE MOON AROUND THE SUN. ONE MAY ASK, HOW MANY TIMES ARE THOSE ORBITS WRAPPED AROUND EACH OTHER (UNTIL THEY END UP AT THE SAME POINTS, IF THEY EVER DO). ASSUME THEY DO AFTER ONE YEAR.

THE NUMBER OF WINDINGS IS CALCULATED WITH THE GAUSS (WHO ELSE?) LINKAGE NUMBER (AN INTEGER), OR LINKING INTEGRAL

$$L_{x,y} = -\frac{1}{4\pi}\oint ds \oint ds' \, \frac{d\vec{x}(s)}{ds} \cdot \left\{ \frac{\vec{r}}{r^3} \times \frac{d\vec{y}(s')}{ds'} \right\}$$

HERE $\vec{x}(s)$ AND $\vec{y}(s')$ ARE THE TWO ORBITS, WHILE $\vec{r} = \vec{x}(s) - \vec{y}(s')$.

WRITING THE VECTOR POTENTIAL IN INTEGRAL FORM ONE FINDS:

$$H = -\frac{1}{4\pi}\int d^3x \int d^3x' \, \vec{B}(\vec{x}) \cdot \left\{ \frac{\vec{r}}{r^3} \times \vec{B}(\vec{x}') \right\}$$

AN ALMOST IDENTICAL EXPRESSION, EXCEPT THE INTEGRAL IS OVER THE VOLUME NOW!