# MHD waves and instabilities

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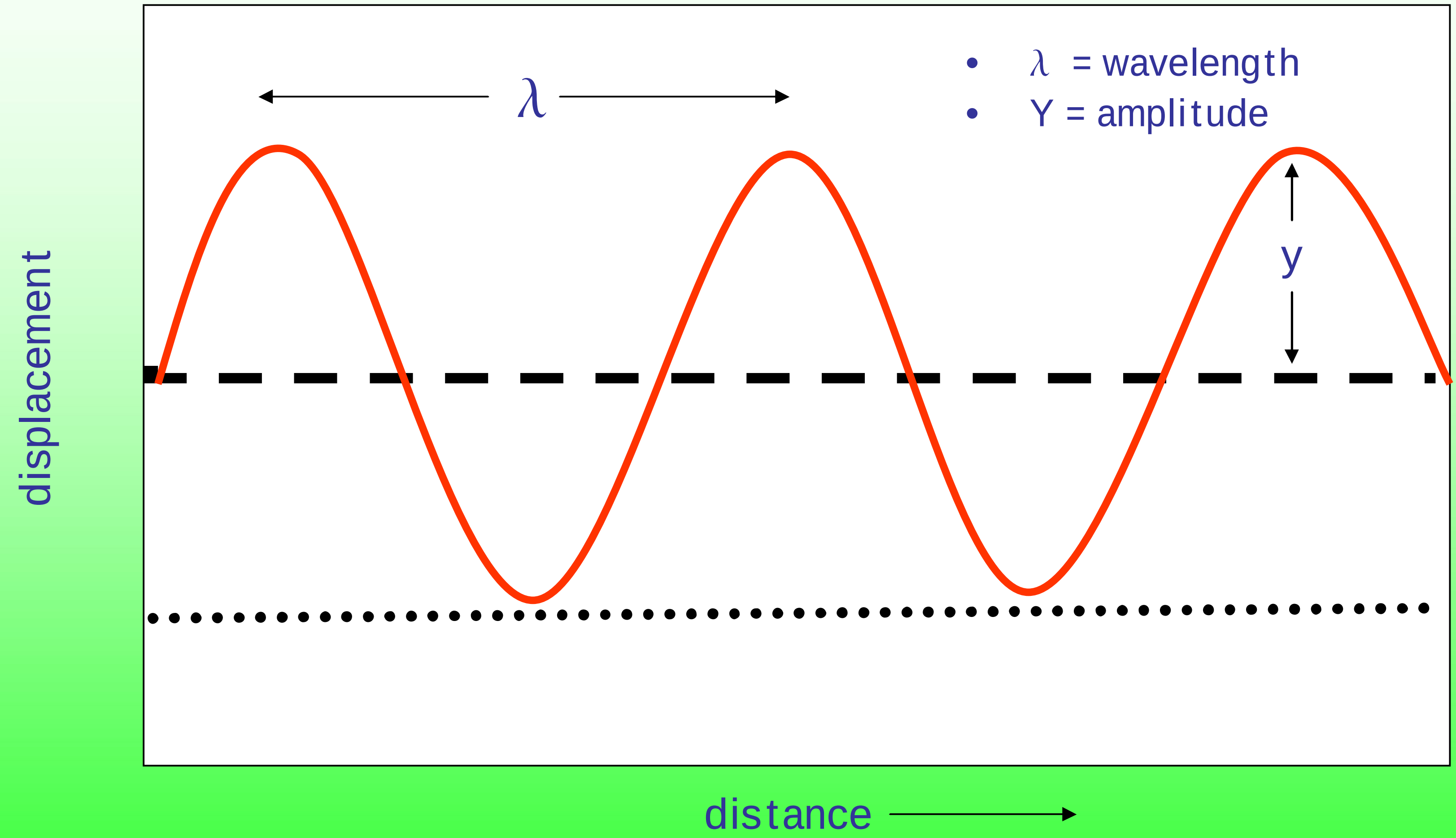
# Wave: a function of time & space

A wave is described by a wave function that is a function of both space and time. If the wave function was sine function then the wave would be expressed by

$$\Phi(x,t) = \bar{A} \sin(\omega t \pm kx)$$

where $\bar{A}$ is the amplitude of the wave, $\omega$ is the angular frequency of the wave and $k$ is the wave number. The negative sign is used for a wave traveling in the positive $x$ direction and the positive sign is used for a wave traveling in the negative $x$ direction.

# Frequency and wavenumber




Frequency: no. of occurrences per unit time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# Phase and Group (velocity) speed

- The **phase velocity** of a wave is the rate at which the phase (of any one frequency) of the wave propagates in space.

The phase velocity is given in terms of the wavelength $\lambda$ (lambda) & period *T* as

$$v_p = \frac{\lambda}{T}. \qquad v_p = \frac{\omega}{k}.$$

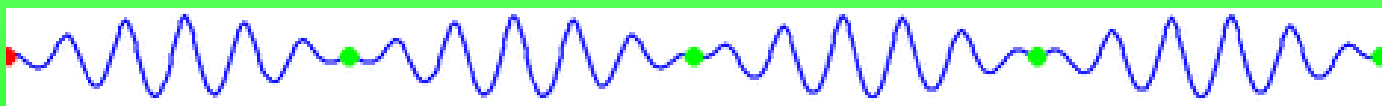
Or, equivalently, in terms of the wave's angular frequency $\omega$ & wavenumber *k*.

The group velocity is velocity of the envelope. It is the velocity at which energy or information is conveyed along a wave. It is defined as

$$v_g \equiv \frac{\partial \omega}{\partial k}$$

If the group velocity is equal to one, the wavepackets move at the same speed as the individual waves. This is true for ordinary sound and light waves. But not necessarily true for surface water waves, quantum electron waves etc.

In a dispersive medium, the phase velocity varies with frequency and is not necessarily the same as the group velocity of the wave, which is the rate that changes in amplitude (known as the *envelope* of the wave) propagate.



● Phase velocity ● Group velocity

# Magnetohydrodynamic (MHD) waves

- Waves are a means to learn about a system by nudging the system and watching how it responds (e.g. pendulum)
- The period of oscillation gives us a relation between the characteristic of the system (e.g. the length of the pendulum, the force trying to restore the equilibrium)
- Similarly a conducting fluid sets up oscillations when disturbed from its equilibrium. The period of oscillation is related to the characteristics of the conducting fluid such as density, pressure, temperature etc.

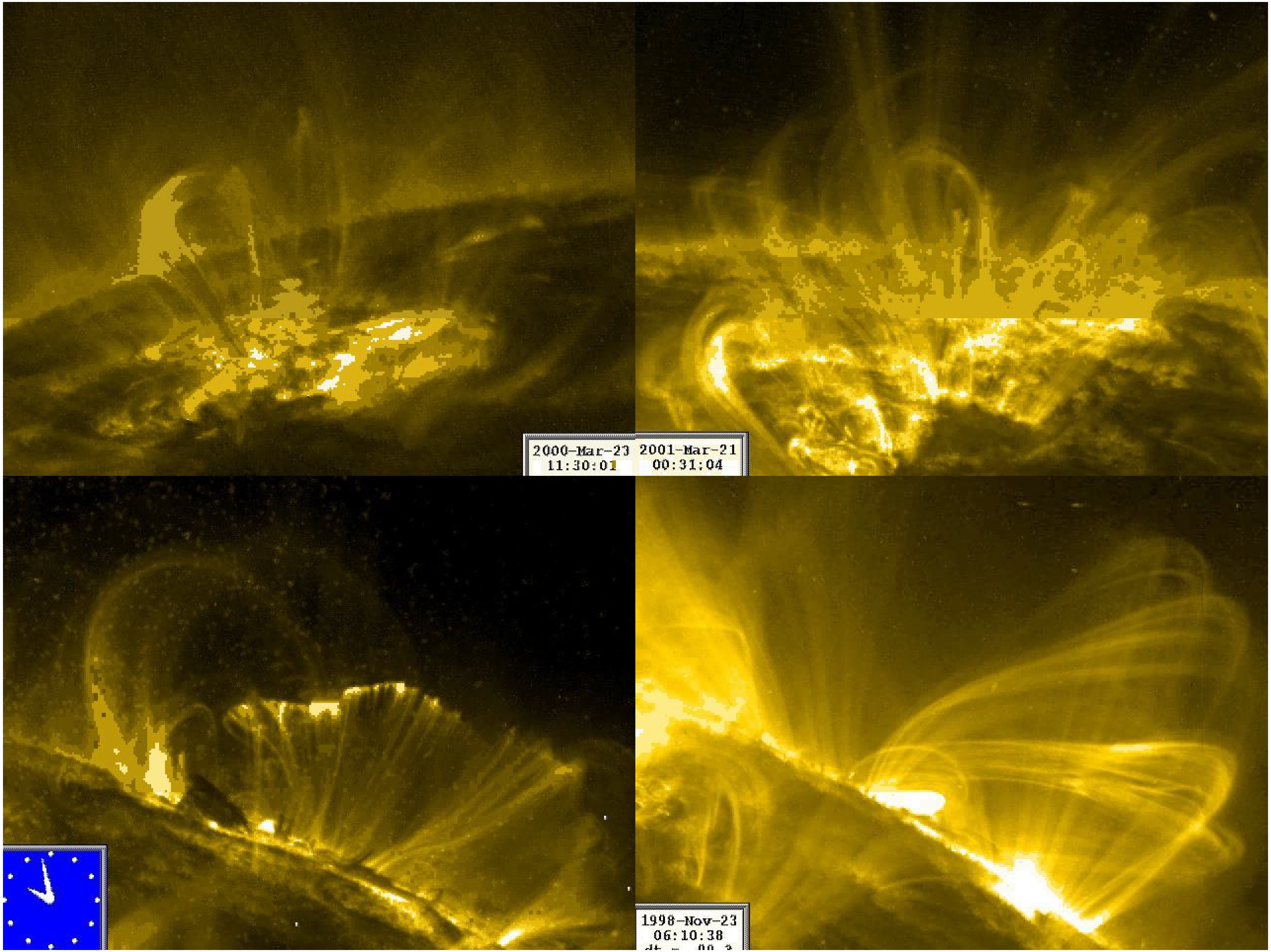
**These oscillations in the presence of a magnetic field are called MHD waves**

# Linear MHD waves

- Linear waves: when the conducting fluid is disturbed by a small amount from its equilibrium.
- Gives information about the phase speed and the group velocity
- The only property we cannot determine is the amplitude of the wave (requires nonlinear wave studies)

# Examples of Magnetohydrodynamic (MHD) waves

- Waves in magnetic flux tubes have recently been observed in the solar atmosphere.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# MHD Equations

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B},$$ Induction

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$ Mass continuity

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g} + \rho \nu \nabla^2 \mathbf{v},$$ Motion

$$\frac{\rho^\gamma}{\gamma - 1} \frac{D}{Dt} \left( \frac{p}{\rho^\gamma} \right) = \nabla \cdot (\kappa \nabla T) - \rho^2 Q(T) + \frac{j^2}{\sigma} + H,$$ Energy

$$p = \frac{\rho \Re T}{\tilde{\mu}},$$ Gas law,

$$\nabla \cdot \mathbf{B} = 0.$$

$$\rho = \rho_0 + \rho_1, \quad p = p_0 + p_1, \quad \mathbf{B} = \mathbf{B_0} + \mathbf{B}_1,$$

$$\mathbf{v}_1 = \mathbf{v}\, e^{(i\mathbf{k}.\mathbf{r} - i\omega t)}, \quad \mathbf{k} = (k_x, k_y, k_z)$$

Linearise the equations

$$\frac{\partial \rho_1}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{v}_1),$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \mathbf{j_1} \times \mathbf{B}_0 + \mathbf{j}_0 \times \mathbf{B}_1 + \rho_1 \mathbf{g},$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0) + \eta \nabla^2 \mathbf{B},$$

$$\frac{\partial p_1}{\partial t} + \mathbf{v}_1 \cdot \nabla p_0 = -\gamma \nabla \cdot \mathbf{v} - \mathrm{L}_1,$$

$$\frac{p_1}{p_0} = \frac{\rho_1}{\rho_0} + \frac{T_1}{T_0},$$

$$\nabla \cdot \mathbf{B}_1 = 0.$$

# Linearised Ideal MHD Equations

Reduce to coupled wave equation

$$\rho_0 \frac{\partial^2 \mathbf{v}_1}{\partial t^2} = \nabla\left(\gamma p_0 \nabla \cdot \mathbf{v}_1 + \mathbf{v}_1 \cdot \nabla p_0\right) + \frac{1}{\mu} \nabla \times \nabla \times \left(\mathbf{v}_1 \times \mathbf{B}_0\right) \times \mathbf{B}_0$$

$$+ \frac{1}{\mu} \nabla \times \mathbf{B}_0 \times \nabla \times \left(\mathbf{v}_1 \times \mathbf{B}_0\right) - \nabla \cdot \left(\rho_0 \mathbf{v}_1\right)\mathbf{g},$$

Or as an eigenvalue problem for the frequency, $e^{i\omega t}$

$$-\rho_0 \omega^2 \mathbf{v}_1 = \nabla\left(\gamma p_0 \nabla \cdot \mathbf{v}_1 + \mathbf{v}_1 \cdot \nabla p_0\right) + \frac{1}{\mu} \nabla \times \nabla \times \left(\mathbf{v}_1 \times \mathbf{B}_0\right) \times \mathbf{B}_0$$

$$+ \frac{1}{\mu} \nabla \times \mathbf{B}_0 \times \nabla \times \left(\mathbf{v}_1 \times \mathbf{B}_0\right) - \nabla \cdot \left(\rho_0 \mathbf{v}_1\right)\mathbf{g},$$

# Sound waves

No $\nu$, $\eta$ or $\mathbf{B}$, no $\mathbf{g}$ => constant $P_0$ and $\rho_0$

$$-\rho_0\omega^2\mathbf{v}_1 = \nabla\left(\gamma p_0 \nabla\cdot\mathbf{v}_1 + \mathbf{v}_1\cdot\nabla p_0\right) + \frac{1}{\mu}\nabla\times\nabla\times\left(\mathbf{v}_1\times\mathbf{B}_0\right)\times\mathbf{B}_0$$

$$+\frac{1}{\mu}\nabla\times\mathbf{B}_0\times\nabla\times\left(\mathbf{v}_1\times\mathbf{B}_0\right) - \nabla\cdot\left(\rho_0\mathbf{v}_1\right)\mathbf{g},$$

# Sound waves

Assume $\mathbf{v}_1 = \mathbf{v}_1 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

$P_0$, $\rho_0$ - both constant

$$-\rho_0\omega^2\mathbf{v}_1 = -\mathbf{k}(\gamma p_0\mathbf{k}\cdot\mathbf{v}_1),$$

$$\Rightarrow \quad \omega^2\mathbf{v}_1 = c_s^2(\mathbf{k}\cdot\mathbf{v}_1)\mathbf{k},$$

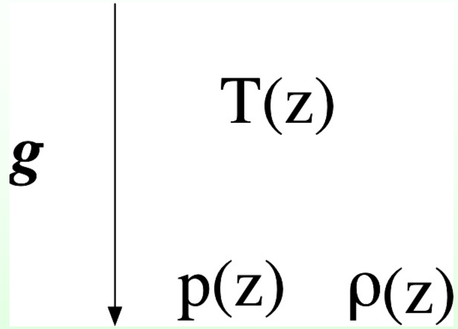
$$\Rightarrow \quad (\omega^2 - k^2c_s^2)(\mathbf{k}\cdot\mathbf{v}_1) = 0,$$

$$\Rightarrow \quad \omega^2 = k^2c_s^2, \qquad (\mathbf{k}\cdot\mathbf{v}_1) \neq 0,$$

Velocity parallel to wavevector, **k**.
Sound waves are compressible.
Phase speed is constant => non dispersive.
Isotropic => no preferred direction.

## No ν, η or B, T(z), => p(z) and ρ(z)

$$\frac{dp_0}{dz} = -\rho_0 g,$$ Equilibrium




## Gravity adds two effects: Cut-off and gravity modes

Assume $\mathbf{v}_1 = \mathbf{v}_1(z) e^{i(k_x x - \omega t)}$, $\quad Q = \rho_0^{1/2} c_s^2 \nabla \cdot \mathbf{v}_1$

$$\Rightarrow \quad \frac{d^2 Q}{dz^2} + K^2(z) Q = 0, \qquad K^2(z) = \frac{\omega^2 - \omega_a^2}{c_s^2} + k_x^2 \left( \frac{\omega_g^2}{\omega^2} - 1 \right)$$

$$\omega_a^2 = \frac{c_s^2}{4H^2}(1 - 2H'),$$ acoustic cut-off frequency and $H = \rho_0 / \rho_0'$

$$\omega_g^2 = -g \left( \frac{g}{c_s^2} + \frac{1}{H} \right),$$ Brunt - Väisälä (buoyancy) frequency.

Oscillates if $K^2(z) > 0$.



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# Basic MHD Waves

We start from the MHD equations expressing the conservation of mass, momentum and energy. We ignore viscosity, gravity, thermal conduction and other nonadiabatic processes.

Continuity Equation

$$\frac{D\rho}{Dt} = -\rho\vec{\nabla}\cdot\vec{v}$$

← Fully Compressible

Momentum Equation

$$\rho\frac{D\vec{v}}{Dt} = -\vec{\nabla}P + \frac{\left((\vec{\nabla}\times\vec{B})\times\vec{B}\right)}{4\pi}$$

$$c_s^2 = \frac{\gamma P}{\rho}$$

Energy Equation

$$\frac{DP}{Dt} = c_s^2\frac{D\rho}{Dt}$$

← Adiabatic

Induction Equation

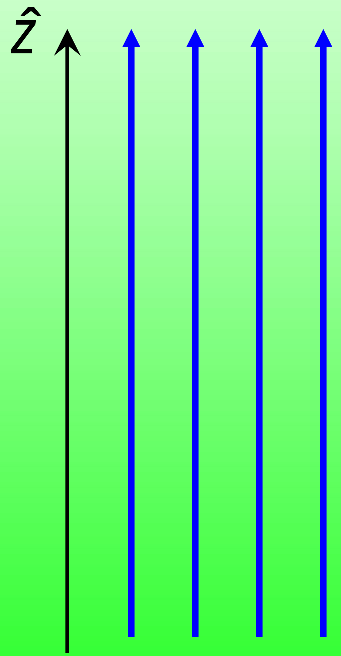
$$\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times\left(\vec{v}\times\vec{B}\right)$$

← Ideal MHD

# Linearize About a Homogeneous Background

Let the background fluid be stationary and homogeneous, with constant density $\rho_0$ and pressure $P_0$ as a function of position.

Further, consider a constant background magnetic field of strength $B_0$, that points in the $z$ direction.




$$\vec{B} = B_0\hat{z} = \text{constant}$$

$$P_0 = \text{constant}$$

$$\rho_0 = \text{constant}$$

Background Media is Homogeneous

$$\vec{B}(\vec{x},t) = \vec{B}_0 + \vec{B}_1(\vec{x},t)$$

$$P(\vec{x},t) = P_0 + P_1(\vec{x},t)$$

$$\rho(\vec{x},t) = \rho_0 + \rho_1(\vec{x},t)$$

$$\vec{v}(\vec{x},t) = \vec{v}_1(\vec{x},t)$$

This subscript will be dropped

# Linearized MHD Equations

Since the atmosphere is homogeneous (without gravitational stratification) and the background magnetic field is constant, the linearized form of the MHD equations is relatively simple.

| | |
|---|---|
| Continuity Equation | $$\frac{\partial \rho_1}{\partial t} = -\rho_0 \vec{\nabla} \cdot \vec{v}$$ |
| Momentum Equation | $$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} P_1 + \frac{(\vec{\nabla} \times \vec{B}_1) \times \vec{B}_0}{4\pi}$$ |
| Energy Equation | $$\frac{\partial P_1}{\partial t} = c_s^2 \frac{\partial \rho_1}{\partial t}$$ |
| Induction Equation | $$\frac{\partial \vec{B}_1}{\partial t} = \vec{\nabla} \times \left( \vec{v} \times \vec{B}_0 \right)$$ |

# Plane Waves

Since the atmosphere is homogeneous, all of the coefficients in the previous set of PDEs are constants. Thus, we should seek plane-wave solutions,

$$\rho_1(\vec{x},t) = \tilde{\rho}_1 \exp\left(i\vec{k}\cdot\vec{x} - i\omega t\right)$$

$$P_1(\vec{x},t) = \tilde{P}_1 \exp\left(i\vec{k}\cdot\vec{x} - i\omega t\right)$$

$$\vec{v}(\vec{x},t) = \tilde{\vec{v}} \exp\left(i\vec{k}\cdot\vec{x} - i\omega t\right)$$

$$\vec{B}_1(\vec{x},t) = \tilde{\vec{B}}_1 \exp\left(i\vec{k}\cdot\vec{x} - i\omega t\right)$$

$\omega$ = Frequency

$\vec{k}$ = Wavenumber

For simplicity, I will drop all of the tildes from here on forward.

# Fourier Transformed Equations

Insert the plane wave function form (or Fourier Transform the equations) to find the following

Continuity Equation

$$-i\omega\rho_1 = -i\rho_0\vec{k}\cdot\vec{v}$$

Momentum Equation

$$-i\omega\rho_0\vec{v} = -i\vec{k}P_1 + \frac{(i\vec{k}\times\vec{B}_1)\times\vec{B}_0}{4\pi}$$

Energy Equation

$$-i\omega P_1 = -i\omega c_s^2\rho_1$$

Induction Equation

$$-i\omega\vec{B}_1 = i\vec{k}\times\left(\vec{v}\times\vec{B}_0\right)$$

# Reduce to a Single Equation

Our goal is to eliminate every variable except the velocity.

We can eliminate the pressure perturbation in favour of the density perturbation through the energy equation

$$-i\omega P_1 = -i\omega c_s^2 \rho_1 \quad \longrightarrow \quad P_1 = c_s^2 \rho_1$$

We can eliminate the density perturbation through the use of the continuity equation

$$-i\omega \rho_1 = -i\vec{k}\rho_0 \vec{v} \quad \longrightarrow \quad \rho_1 = \frac{\rho_0}{\omega}\vec{k}\cdot\vec{v}$$

The induction equation can be used to eliminate the perturbed magnetic field

$$-i\omega \vec{B}_1 = i\vec{k}\times\left(\vec{v}\times\vec{B}_0\right) \quad \longrightarrow \quad \vec{B}_1 = -\frac{\vec{k}}{\omega}\times\left(\vec{v}\times\vec{B}_0\right)$$

# The momentum equation now is

$$-i\omega\rho_0\vec{v} = -i\vec{k}P_1 + \frac{(i\vec{k} \times \vec{B}_1) \times \vec{B}_0}{4\pi}$$

We can substitute $P_1$ from the energy equation and $B_1$ from the induction equation and get a single equation for v.

# Alfvén Velocity

$$P_1 = c_s^2 \rho_1$$

$$\rho_1 = \frac{\rho_0}{\omega} \vec{k} \cdot \vec{v}$$

$$\vec{B}_1 = -\frac{\vec{k}}{\omega} \times \left( \vec{v} \times \vec{B}_0 \right)$$

$$-i\omega\rho_0\vec{v} = -i\vec{k}P_1 + \frac{(i\vec{k} \times \vec{B}_1) \times \vec{B}_0}{4\pi}$$

Alfvén Velocity

$$\vec{V}_A = \frac{\vec{B}_0}{\sqrt{4\pi\rho_0}}$$

$$\left[\omega^2 - (\vec{k} \cdot \vec{V}_A)^2\right]\vec{v} = \left[\left(c_s^2 + V_A^2\right)(\vec{k} \cdot \vec{v}) - (\vec{k} \cdot \vec{V}_A)(\vec{V}_A \cdot \vec{v})\right]\vec{k} - (\vec{k} \cdot \vec{V}_A)(\vec{k} \cdot \vec{v})\vec{V}_A$$

# Simplify

$$\left[\omega^2 - (\vec{k}\cdot\vec{V}_A)^2\right]\vec{v} = \left[\left(c_s^2 + V_A^2\right)(\vec{k}\cdot\vec{v}) - (\vec{k}\cdot\vec{V}_A)(\vec{V}_A\cdot\vec{v})\right]\vec{k} - (\vec{k}\cdot\vec{V}_A)(\vec{k}\cdot\vec{v})\vec{V}_A$$

Remember that the background magnetic field points in the ***z*** direction.

$$\left.\begin{aligned}\vec{B}_0 &= B_0\hat{z}\\ \vec{V}_A &= V_A\hat{z}\end{aligned}\right\} \longrightarrow \left\{\begin{aligned}\vec{k}\cdot\vec{V}_A &= k_z V_A\\ \vec{V}_A\cdot\vec{v} &= V_A v_z\end{aligned}\right.$$

We can further simplify be noting that ***x*** and ***y*** are interchangeable. Therefore, without loss of generality we may assume $\boldsymbol{k_y = 0}$.

$$\left(\omega^2 - k_z^2 V_A^2\right)\vec{v} = \left[\left(c_s^2 + V_A^2\right)(\vec{k}\cdot\vec{v}) - k_z V_A^2 v_z\right]\vec{k} - k_z V_A^2(\vec{k}\cdot\vec{v})\hat{z}$$

# Matrix Formulation

$$\left(\omega^2 - k_z^2 V_A^2\right)\vec{v} = \left[\left(c_s^2 + V_A^2\right)(\vec{k}\cdot\vec{v}) - k_z V_A^2 v_z\right]\vec{k} - k_z V_A^2 (\vec{k}\cdot\vec{v})\hat{z}$$

This equation is actually three separate equations, one for each component.
Those three equations are coupled and can be written in a matrix form. $\mathbb{A}\vec{v} = -\omega^2\vec{v}$

$$\mathbb{A} = \begin{pmatrix} -k_z^2 V_A^2 - k_x^2\left(c_s^2 + V_A^2\right) & 0 & -k_x k_z c_s^2 \\ 0 & -k_z^2 V_A^2 & 0 \\ -k_x k_z c_s^2 & 0 & -k_z^2 c_s^2 \end{pmatrix}$$

Zeros because $\boldsymbol{k_y} = \mathbf{0}$

# Eigenproblem

$$\mathbb{A}\vec{V} = -\omega^2\vec{V}$$

**This is an eigenvalue-eigenvector problem**

- Since the matrix is 3x3, there are three eigenvalues and three eigenvectors. Each corresponds to a separate wave mode.
- The three eigenvalues $\omega^2$ provide the dispersion relations.
- The eigenvectors provide the polarizations.
- The eigenvectors are orthogonal, and any disturbance can be expressed as a linear combination of the three wave modes.

# Dispersion Relation - Eigenvalues

$$\left(\mathbb{A} + \omega^2 \mathbb{I}\right)\vec{v} = 0$$

If this matrix equation is to have a solution, the determinant of the matrix must vanish.

$$\det\left(\mathbb{A} + \omega^2 \mathbb{I}\right) = 0$$

After some algebra we obtain the dispersion relation

$$\left(\omega^2 - k_z^2 V_A^2\right)\left[\omega^4 - k^2\left(c_s^2 + V_A^2\right)\omega^2 + k^2 k_z^2 c_s^2 V_A^2\right] = 0$$

# Three Wave Modes

$$\left(\omega^2 - k_z^2 V_A^2\right)\left[\omega^4 - k^2\left(c_s^2 + V_A^2\right)\omega^2 + k^2 k_z^2 c_s^2 V_A^2\right] = 0$$

This equation is cubic in $\omega^2$. Thus, there are three unique solutions for $\omega^2$, and correspondingly three unique wave modes.

One solution satisfies

$$\omega^2 - k_z^2 V_A^2 = 0$$

Alfvén Wave

Two solutions satisfy

$$\omega^4 - k^2\left(c_s^2 + V_A^2\right)\omega^2 + k^2 k_z^2 c_s^2 V_A^2 = 0$$

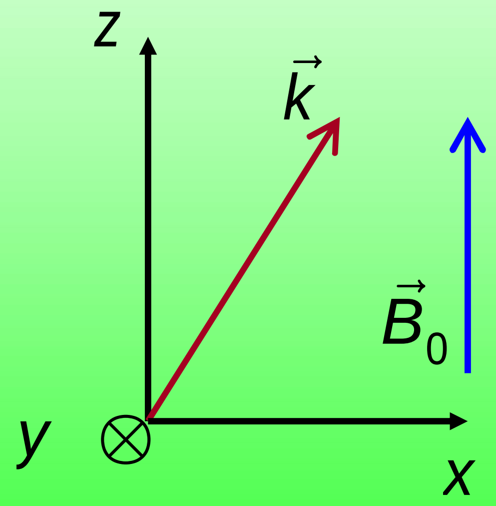
Fast and Slow Magnetoacoustic Waves

$$\omega^2 = \frac{k^2}{2}\left(c_s^2 + V_A^2\right) \pm \frac{k^2}{2}\sqrt{\left(c_s^2 + V_A^2\right)^2 - 4\frac{k_z^2}{k^2} c_s^2 V_A^2}$$

# Polarizations - Eigenvectors

$$\mathbb{A} = \begin{pmatrix} -k_z^2 V_A^2 - k_x^2\left(c_s^2 + V_A^2\right) & 0 & -k_x k_z c_s^2 \\ 0 & -k_z^2 V_A^2 & 0 \\ -k_x k_z c_s^2 & 0 & -k_z^2 c_s^2 \end{pmatrix}$$

The three eigenvectors give the solution for the velocity for each wave mode.




The Alfvén wave is polarized in the ***y*** direction

$$\vec{v} = U_A \hat{y}$$

The magnetoacoustic waves have polarization in the ***x-z*** plane.

$$\vec{v} = U_{f,s}\left[\left(\omega_{f,s}^2 - k_z^2 c_s^2\right)\hat{x} + \left(k_x k_z c_s^2\right)\hat{z}\right]$$

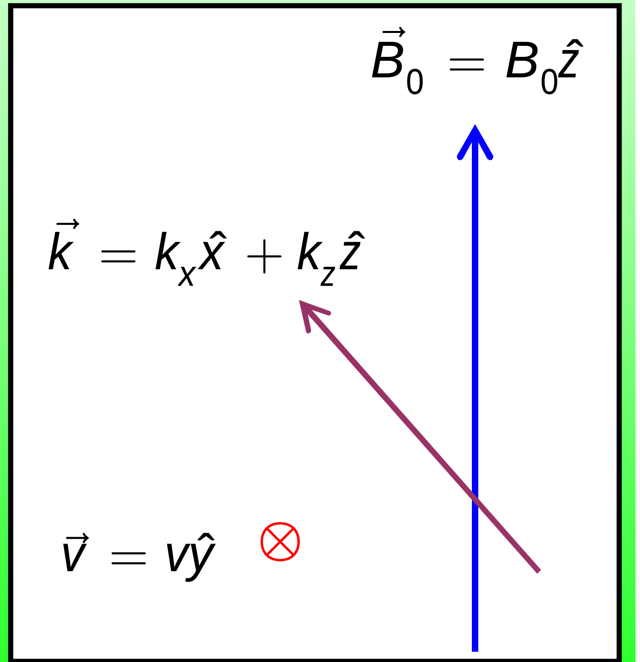
# Shear Alfvén Wave

The Shear Alfvén wave satisfies the dispersion relation.

$$\omega^2 - k_z^2 V_A^2 = 0$$

The polarization of the eigenvector is purely in the ***y*** direction, perpendicular to both the magnetic field and the wavevector.

$$\vec{v} = U_A \hat{y}$$




Clearly the wave is incompressive.

$$\vec{\nabla} \cdot \vec{v} = i\vec{k} \cdot \vec{v} = k_y v_y = 0$$

$$\left.\begin{aligned} \rho_1 &= 0 \\ P_1 &= 0 \end{aligned}\right\}$$

see previous slide on momentum equation!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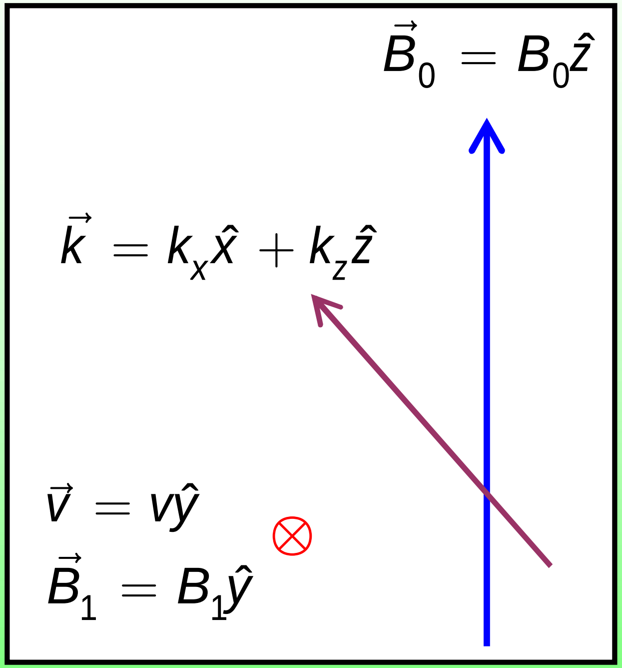
# Alfvén Waves are Transverse

The perturbed magnetic field is also purely in the ***y*** direction. This can be shown using the induction equation.

$$\vec{B}_1 = -\frac{\vec{k}}{\omega} \times \left( \vec{v} \times \vec{B}_0 \right)$$




$$\vec{v} = U_A\hat{y}$$

$$\vec{B}_1(\vec{x},t) = -\frac{k_z B_0}{\omega} U_A \hat{y}$$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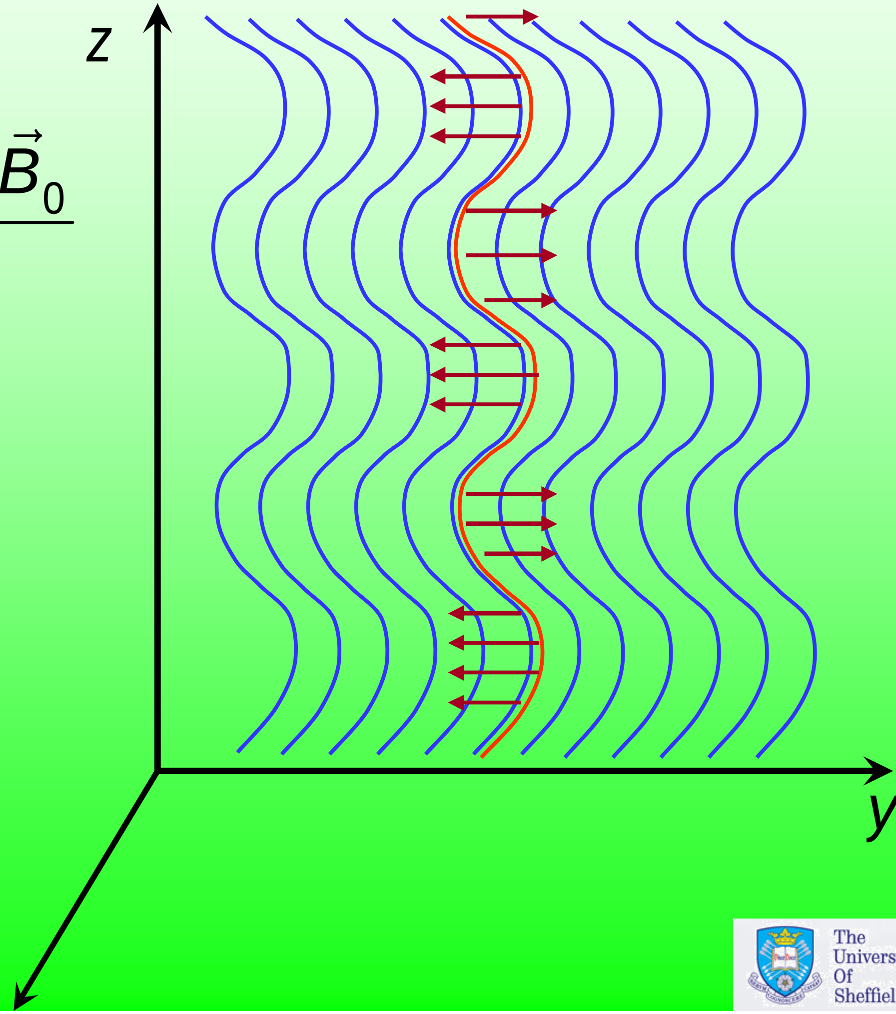
# Alfvén Waves are Tension Waves

Since Alfvén waves are incompressive, they lack perturbations to the magnetic pressure and the gas pressure. Thus, the restoring force must be magnetic tension.

$$-i\omega\rho_0\vec{v} = -i\vec{k}P_1 + \frac{(i\vec{k}\times\vec{B}_1)\times\vec{B}_0}{4\pi}$$




The tension force

$$\frac{(i\vec{k}\times\vec{B}_1)\times\vec{B}_0}{4\pi} = \frac{ik_zB_1B_0}{4\pi}\hat{y}$$

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# Magnetoacoustic Waves

The two magnetoacoustic waves satisfy the dispersion relation

$$\omega^4 - k^2\left(c_s^2 + V_A^2\right)\omega^2 + k^2 k_z^2 c_s^2 V_A^2 = 0$$

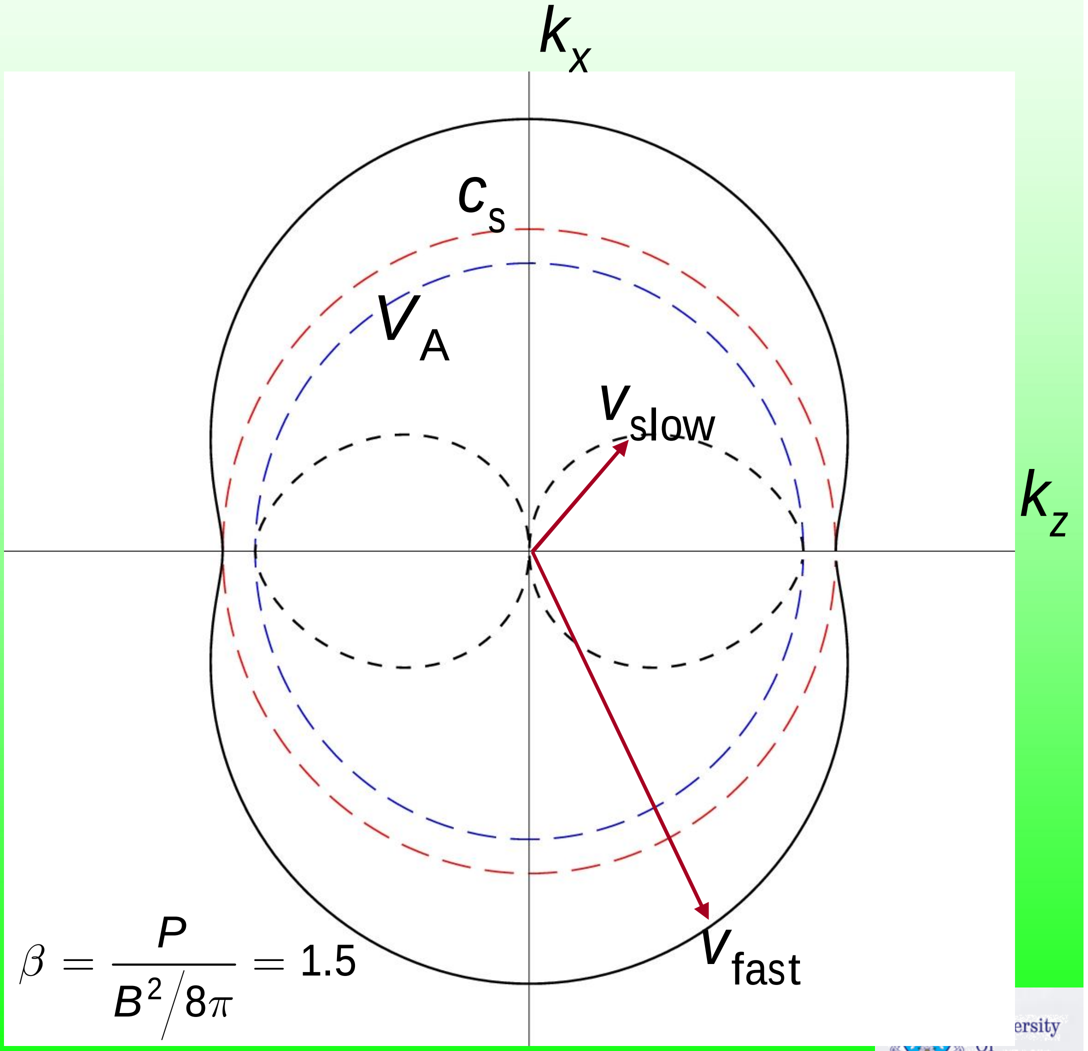
Quadratic equation in $\omega^2$

$$\omega^2 = \frac{k^2}{2}\left(c_s^2 + V_A^2\right) \pm \frac{k^2}{2}\sqrt{\left(c_s^2 + V_A^2\right)^2 - 4\frac{k_z^2}{k^2}c_s^2 V_A^2}$$

The phase speed is obtained by dividing by the wave number.

$$v_{\text{phase}}^2 = \left(\frac{\omega}{k}\right)^2 = \frac{1}{2}\left\{\left(c_s^2 + V_A^2\right) \pm \sqrt{\left(c_s^2 + V_A^2\right)^2 - 4\frac{k_z^2}{k^2}c_s^2 V_A^2}\right\}$$

$$v_{\text{phase}}^2 = \frac{1}{2}\left\{\left(c_s^2 + V_A^2\right) \pm \sqrt{\left(c_s^2 + V_A^2\right)^2 - 4\frac{k_z^2}{k^2}c_s^2V_A^2}\right\}$$




Magnetoacoustic Modes

**+** sign → Fast mode

**–** sign → Slow mode

The fast mode propagates faster than either $\boldsymbol{c_s}$ or $\boldsymbol{V_A}$.

The slow mode propagates slower than either $\boldsymbol{c_s}$ or $\boldsymbol{V_A}$.

# Tube Speed or Cusp Speed

$$v_{\text{phase}}^2 = \frac{1}{2}\left\{\left(c_s^2 + V_A^2\right) \pm \sqrt{\left(c_s^2 + V_A^2\right)^2 - 4\frac{k_z^2}{k^2}c_s^2 V_A^2}\right\}$$

This equation can be expressed in a useful form using the cusp speed

$$v_{\text{phase}}^2 = \frac{c_s^2 + V_A^2}{2}\left[1 \pm \sqrt{1 - 4\left(\frac{k_z}{k}\right)^2 \frac{c_T^2}{c_s^2 + V_A^2}}\right]$$

$$c_T^2 = \frac{c_s^2 V_A^2}{c_s^2 + V_A^2}$$

Tube Speed
Cusp Speed
Slow Speed

$$\frac{1}{c_T^2} = \frac{1}{c_s^2} + \frac{1}{V_A^2}$$



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# Plasma $\beta$-parameter

$$c_T^2 = \frac{c_s^2 V_A^2}{c_s^2 + V_A^2}$$

The tube speed is small if either the sound speed or the Alfvén speed are small compared to the other. This can be expressed through the plasma's $\beta$-parameter.

$$\beta = \frac{8\pi P}{B^2} = \frac{2}{\gamma}\frac{c_s^2}{V_A^2}$$

small in either limit

If $\beta \ll 1$ then $c_s^2 \ll V_A^2$

$$c_T^2 \to c_s^2$$

$$\frac{c_T^2}{c_s^2 + V_A^2} \to \frac{c_s^2}{V_A^2} \ll 1$$

If $\beta \gg 1$ then $V_A^2 \ll c_s^2$

$$c_T^2 \to V_A^2$$

$$\frac{c_T^2}{c_s^2 + V_A^2} \to \frac{V_A^2}{c_s^2} \ll 1$$

# Phase Speed Limits

$$v_{\text{phase}}^2 = \frac{c_s^2 + V_A^2}{2}\left[1 \pm \sqrt{1 - 4\left(\frac{k_z}{k}\right)^2 \frac{c_T^2}{c_s^2 + V_A^2}}\right]$$

If either the sound speed or Alfvén speed are much larger than the other, the square root term may be simplified.

$$v_{\text{phase}}^2 = \frac{c_s^2 + V_A^2}{2}\left[1 \pm \left(1 - 2\left(\frac{k_z}{k}\right)^2 \frac{c_T^2}{c_s^2 + V_A^2}\right)\right]$$

$$\left(\frac{\omega}{k}\right)^2 = c_s^2 + V_A^2 - \left(\frac{k_z}{k}\right)^2 c_T^2$$

Fast Mode

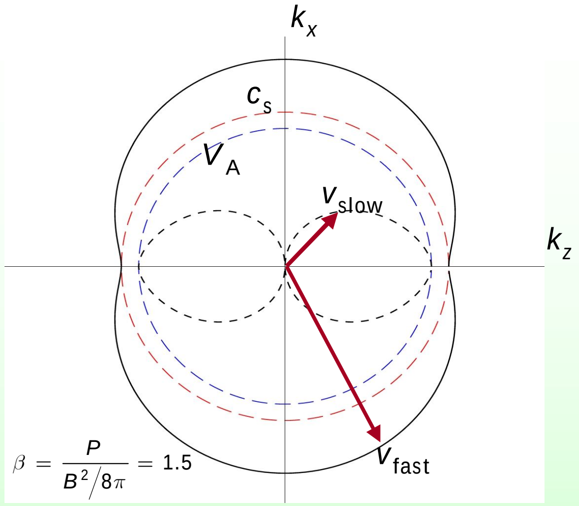
$$\left(\frac{\omega}{k}\right)^2 = \left(\frac{k_z}{k}\right)^2 c_T^2$$

Slow Mode

# Fast Mode

$$\left(\frac{\omega}{k}\right)^2 = c_s^2 + V_A^2 - \left(\frac{k_z}{k}\right)^2 c_T^2$$

$$\vec{v} = U\left[\left(\omega^2 - k_z^2 c_s^2\right)\hat{x} + \left(k_x k_z c_s^2\right)\hat{z}\right]$$




In the limit of weak magnetic field, the fast mode is acoustic in nature with a weak magnetic correction

$$\beta \gg 1$$

$$c_s^2 \gg V_A^2$$

$$\left(\frac{\omega}{k}\right)^2 = c_s^2 + \left(\frac{k_x}{k}\right)^2 V_A^2$$

$$\vec{v} = U\hat{k}$$

In the limit of strong magnetic field, the fast mode is driven largely by magnetic pressure and tension.

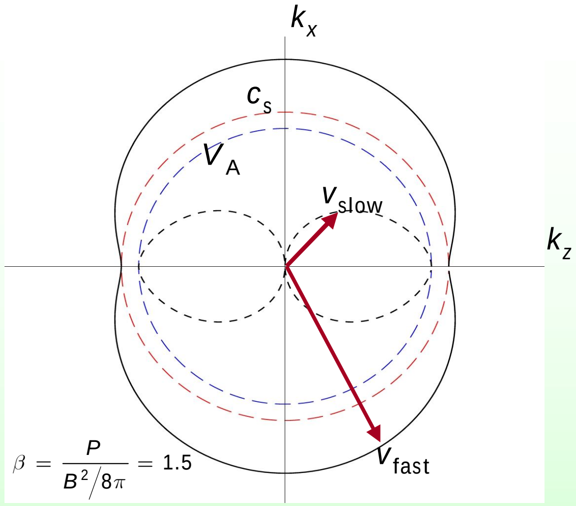
$$\beta \ll 1$$

$$c_s^2 \ll V_A^2$$

$$\left(\frac{\omega}{k}\right)^2 = V_A^2 + \left(\frac{k_x}{k}\right)^2 c_s^2$$

$$\vec{v} = U\left[\left(k^2 V_A^2\right)\hat{x} + \left(k_x k_z c_s^2\right)\hat{z}\right]$$

# Slow Mode




$$\left(\frac{\omega}{k}\right)^2 = \left(\frac{k_z}{k}\right)^2 c_T^2$$

$$\vec{v} = U\left[\left(\omega^2 - k_z^2 c_s^2\right)\hat{x} + \left(k_x k_z c_s^2\right)\hat{z}\right]$$

In the limit of weak magnetic field, the slow mode is largely a tension wave and behaves much like the Alfvén wave.

$$\beta \gg 1$$

$$c_s^2 \gg V_A^2$$

$$\left(\frac{\omega}{k}\right)^2 = \left(\frac{k_z}{k}\right)^2 V_A^2$$

$$\vec{v} = -U\hat{y} \times \hat{k}$$

In the limit of strong magnetic field, the slow mode is largely acoustic in nature. However, the wave only propagates along field lines.

$$\beta \ll 1$$

$$c_s^2 \ll V_A^2$$

$$\left(\frac{\omega}{k}\right)^2 = \left(\frac{k_z}{k}\right)^2 c_s^2$$

$$\vec{v} = U\hat{z}$$

# Parallel Propagation

If the wave is propagating purely parallel to the magnetic field

$$k_x = 0$$

$$\omega^2 = \frac{k^2}{2}\left(c_s^2 + V_A^2\right) \pm \frac{k^2}{2}\sqrt{\left(c_s^2 + V_A^2\right)^2 - 4\frac{k_z^2}{k^2}c_s^2 V_A^2}$$

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}\left(c_s^2 + V_A^2\right) \pm \frac{1}{2}\sqrt{\left(c_s^2 + V_A^2\right)^2 - 4c_s^2 V_A^2}$$

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}\left(c_s^2 + V_A^2\right) \pm \frac{1}{2}\left(c_s^2 - V_A^2\right)$$

$$\boxed{\left(\frac{\omega}{k}\right)^2 = c_s^2 \text{ or } V_A^2}$$

Sound Wave and an Alfvén Wave

# Perpendicular Propagation

If the wave is propagating purely perpendicular to the magnetic field

$$k_z = 0$$

$$\omega^2 = \frac{k^2}{2}\left(c_s^2 + V_A^2\right) \pm \frac{k^2}{2}\sqrt{\left(c_s^2 + V_A^2\right)^2 - 4\frac{k_z^2}{k^2}c_s^2 V_A^2}$$

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}\left(c_s^2 + V_A^2\right) \pm \frac{1}{2}\sqrt{\left(c_s^2 + V_A^2\right)^2}$$

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}\left(c_s^2 + V_A^2\right) \pm \frac{1}{2}\left(c_s^2 + V_A^2\right)$$

$$\boxed{\left(\frac{\omega}{k}\right)^2 = 0 \ \text{ or } \ c_s^2 + V_A^2}$$

Magnetoacoustic Pressure Wave

# Magnetoacoustic Waves are Pressure and Tension Waves

**Fast Mode**

The fast mode is fast because the pressure and tension are nearly in phase.

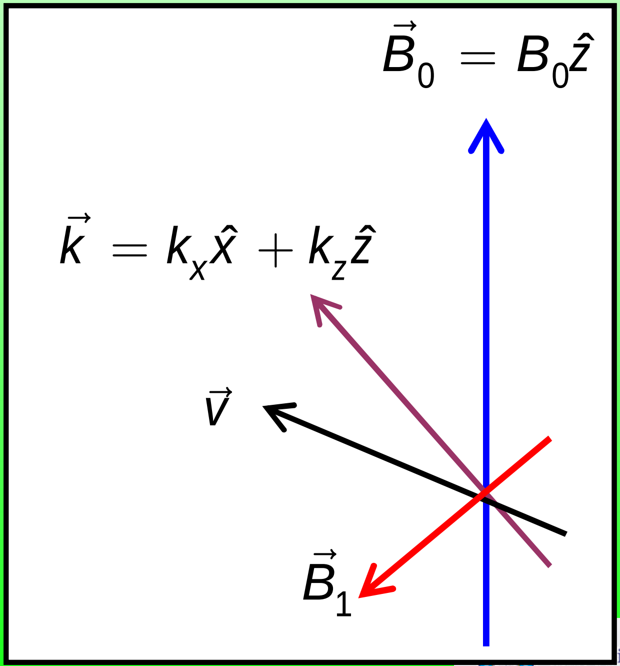
**Slow Mode**

The slow mode is slow because the tension and pressure are nearly out of phase

Only Clean Statements

$$v_y = 0$$

$$B_{1y} = 0$$

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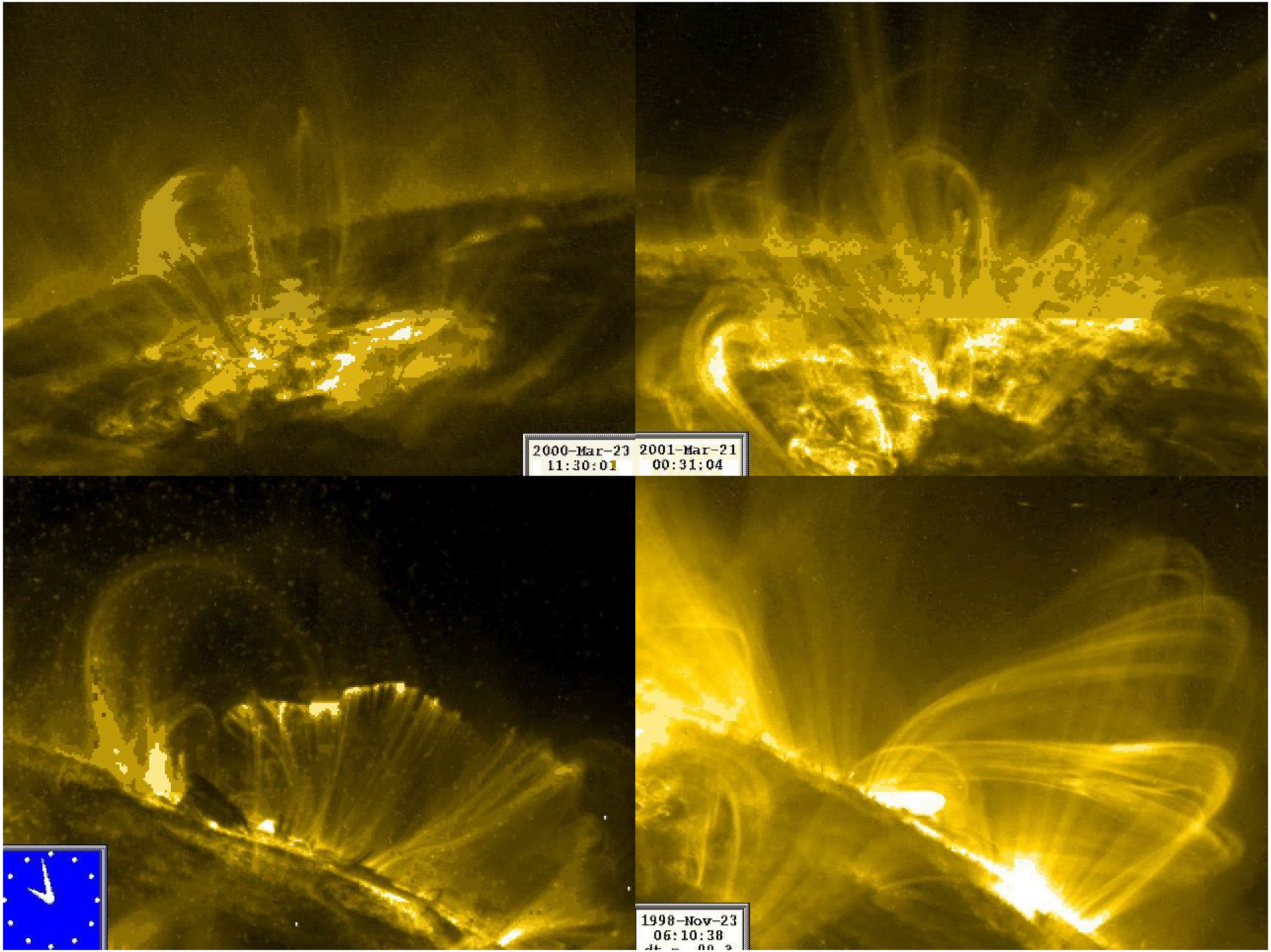
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11:30:01
2001-Mar-21
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1998-Nov-23
06:10:38

# Alfvén Waves in the Corona

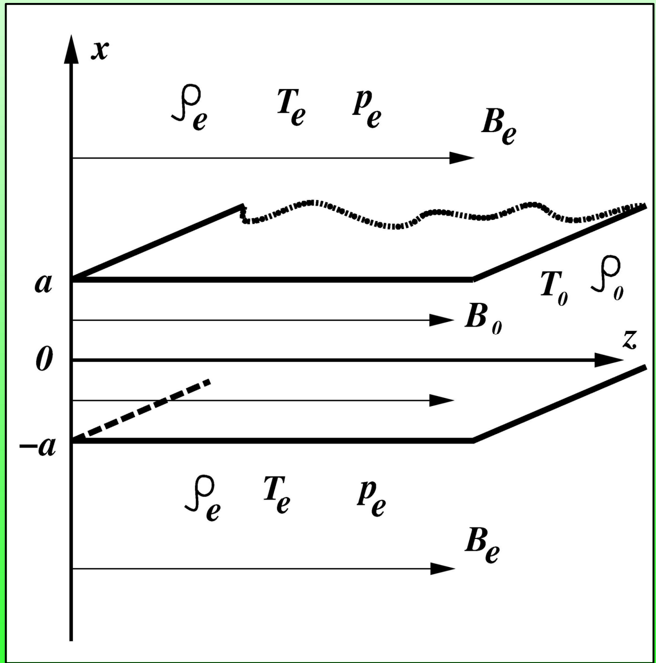
s4.mov

Basic Theory

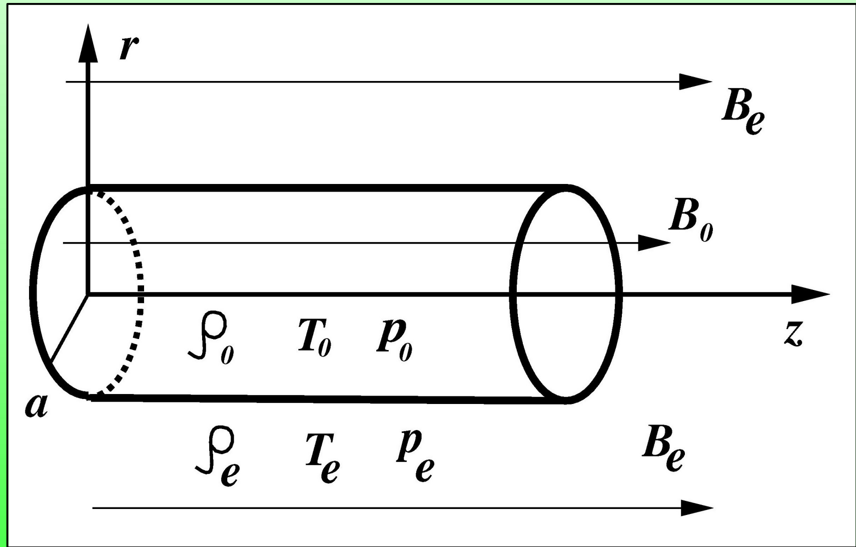
## Structured magnetic fields: Definitions

Coronal fields: high density, low $V_A$

Magnetic slab



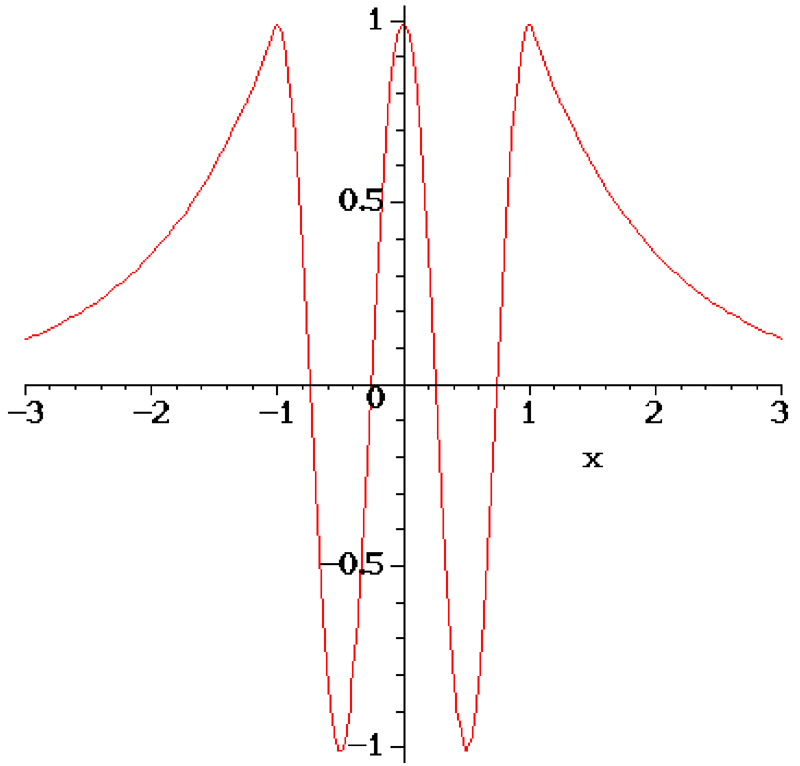

Magnetic tube



# Types of modes: Surface or body

Surface mode:
Disturbance confined to interface

Body mode:
Oscillatory inside tube



# Tube Waves

If the tube is axisymmetric and steady, the azimuthal angle $\phi$ and the time ***t*** are separable variables

$$\vec{v}(r,\varphi,z,t) = \sum_m \vec{u}(r,z)e^{im\varphi}e^{-i\omega t}$$

We are seeking waves on thin tubes that lack internal structure. Only three wave components can possibly satisfy this criterion.

Torsional Alfvén Wave ***m* = 0**

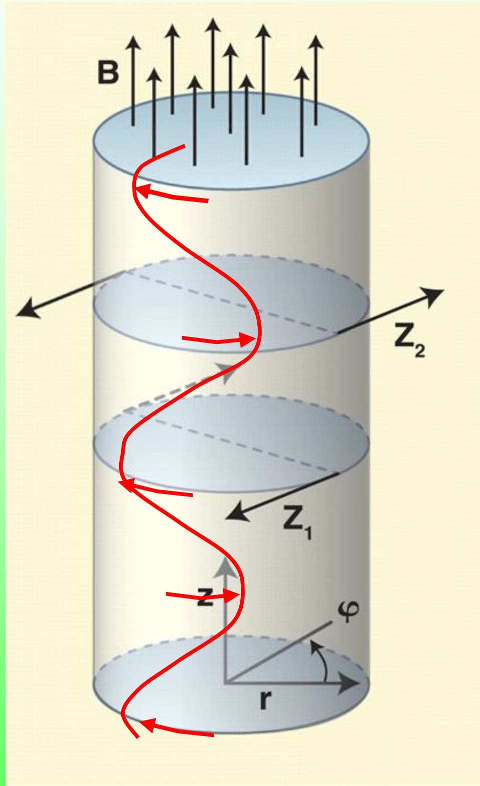
Sausage Wave (or Mode) ***m* = 0**

Kink Wave (or Mode) ***m* = ±1**

# Tube Waves

$m = 0$

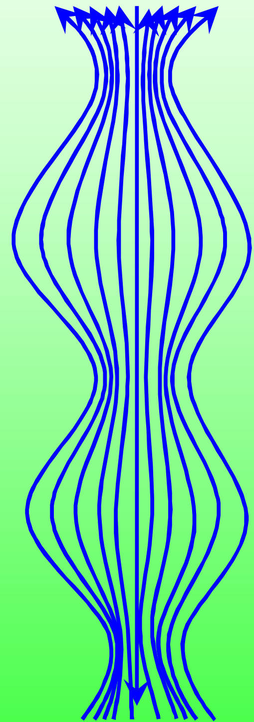
$$\vec{v}(r,\varphi,z,t) = \sum_m \vec{u}(r,z)e^{im\varphi}e^{-i\omega t}$$




Torsional Alfven Wave
Pure Rotation

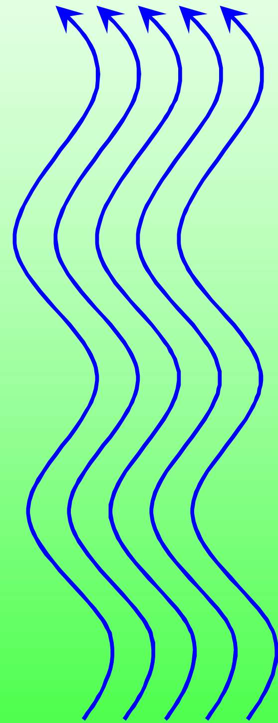
Fedun+Erdelyi

$m = 0$



Sausage Mode
Magnetic axis does not move

$m = \pm 1$



Kink Mode
Magnetic axis moves

# The Wave Equations

We start with the adiabatic ideal MHD equations

$$\frac{D\rho}{Dt} = -\rho \vec{\nabla} \cdot \vec{v}$$ Continuity

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla}\left(P + \frac{B^2}{8\pi}\right) + \frac{\left(\vec{B} \cdot \vec{\nabla}\right)\vec{B}}{4\pi} + \rho \vec{g}$$ Momentum

$$\frac{DP}{Dt} = c_s^2 \frac{D\rho}{Dt} = -\rho c_s^2 \vec{\nabla} \cdot \vec{v}$$ Internal Energy

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B}\right)$$ Induction

# Alternate Form of the Induction Equation

Use the vector identity

$$\vec{\nabla} \times \left(\vec{a} \times \vec{b}\right) = \left(\vec{b} \cdot \vec{\nabla}\right)\vec{a} - \left(\vec{a} \cdot \vec{\nabla}\right)\vec{b} + \vec{a}\left(\vec{\nabla} \cdot \vec{b}\right) - \vec{b}\left(\vec{\nabla} \cdot \vec{a}\right)$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B}\right)$$

$$\frac{\partial \vec{B}}{\partial t} = \left(\vec{B} \cdot \vec{\nabla}\right)\vec{v} - \left(\vec{v} \cdot \vec{\nabla}\right)\vec{B} + \vec{v}\left(\vec{\nabla} \cdot \vec{B}\right) - \vec{B}\left(\vec{\nabla} \cdot \vec{v}\right)$$

0

$$\frac{D\vec{B}}{Dt} = -\vec{B}\left(\vec{\nabla} \cdot \vec{v}\right) + \left(\vec{B} \cdot \vec{\nabla}\right)\vec{v}$$

The field changes due to transverse compression

# The Adiabatic Ideal MHD Equations

$$\frac{D\rho}{Dt} = -\rho \vec{\nabla} \cdot \vec{v}$$

Continuity

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla}\left(P + \frac{B^2}{8\pi}\right) + \frac{\left(\vec{B} \cdot \vec{\nabla}\right)\vec{B}}{4\pi} + \rho\vec{g}$$

Momentum

$$\frac{DP}{Dt} = -\rho c_s^2 \vec{\nabla} \cdot \vec{v}$$

Internal Energy

$$\frac{D\vec{B}}{Dt} = -\vec{B}\left(\vec{\nabla} \cdot \vec{v}\right) + \left(\vec{B} \cdot \vec{\nabla}\right)\vec{v}$$

Induction

# Linearized Equations

Inside the flux tube, the following linearized perturbation equations hold.

$$\frac{\partial \rho_1}{\partial t} + v_z \frac{d\rho_0}{dz} = -\rho_0 \vec{\nabla} \cdot \vec{v}$$

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla}\left(P_1 + \frac{\vec{B}_0 \cdot \vec{B}_1}{4\pi}\right) + \frac{\left(\vec{B}_0 \cdot \vec{\nabla}\right)\vec{B}_1 + \left(\vec{B}_1 \cdot \vec{\nabla}\right)\vec{B}_0}{4\pi} + \rho_1 \vec{g}$$

$$\frac{\partial P_1}{\partial t} + v_z \frac{dP_0}{dz} = -\rho_0 c_s^2 \vec{\nabla} \cdot \vec{v}$$

$$\frac{\partial \vec{B}_1}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right)\vec{B}_0 = -\vec{B}_0\left(\vec{\nabla} \cdot \vec{v}\right) + \left(\vec{B}_0 \cdot \vec{\nabla}\right)\vec{v}$$

# Torsional Alfvén Wave

The torsional Alfvén wave consists of purely axisymmetric twisting motions.

$$\vec{v}(r,\phi,z,t) = v(r,z,t)\hat{\phi}$$

Since the motions are axisymmetric all other variables are axisymmetric

$$\frac{\partial v}{\partial \phi} = \frac{\partial \rho_1}{\partial \phi} = \frac{\partial P_1}{\partial \phi} = \frac{\partial B_1}{\partial \phi} = 0$$

This axisymmetry results in an incompressive wave

$$\boxed{\vec{\nabla} \cdot \vec{v} = \frac{1}{r}\frac{\partial v}{\partial \phi} = 0}$$

# Torsional Alfvén Wave

The continuity and energy equation are satisfied trivially

$$\frac{\partial \rho_1}{\partial t} = -v_z \frac{d\rho_0}{dz} - \rho_0 \vec{\nabla} \cdot \vec{v} = 0 \longrightarrow \rho_1 = 0$$

$$\frac{\partial P_1}{\partial t} = -v_z \frac{dP_0}{dz} - \rho_0 c_s^2 \vec{\nabla} \cdot \vec{v} = 0 \longrightarrow P_1 = 0$$

From the induction equation we can see that the perturbed field is toroidal

$$\frac{\partial \vec{B}_1}{\partial t} = -\left(\vec{v} \cdot \vec{\nabla}\right)\vec{B}_0 - \vec{B}_0\left(\vec{\nabla} \cdot \vec{v}\right) + \left(\vec{B}_0 \cdot \vec{\nabla}\right)\vec{v}$$

$$\frac{\partial \vec{B}_1}{\partial t} = -\frac{v}{r}\frac{\partial \vec{B}_0}{\partial \phi} + \left(\vec{B}_0 \cdot \vec{\nabla}\right)\left(v\hat{\phi}\right)$$

# It's a Tension Wave

Note the magnetic pressure is identically zero because the flux tube field is poloidal while the perturbed wave field is toroidal

$$\frac{\vec{B}_0 \cdot \vec{B}_1}{4\pi} = \frac{\left(B_{0r}\hat{r} + B_{0z}\hat{z}\right) \cdot B_1\hat{\phi}}{4\pi} = 0$$

Since we showed earlier that the perturbed gas pressure $\boldsymbol{P_1}$ and density $\boldsymbol{\rho_1}$ are zero, the only restoring force is tension.

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = \frac{\left(\vec{B}_0 \cdot \vec{\nabla}\right)\vec{B}_1 + \left(\vec{B}_1 \cdot \vec{\nabla}\right)\vec{B}_0}{4\pi}$$

# The Restoring Force is Torsional

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = \frac{1}{4\pi}\left[\left(\vec{B}_0 \cdot \vec{\nabla}\right)\vec{B}_1 + \left(\vec{B}_1 \cdot \vec{\nabla}\right)\vec{B}_0\right]$$

$$\vec{B}_1 = B_1\hat{\phi}$$

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = \hat{\phi}\frac{1}{4\pi}\left(\vec{B}_0 \cdot \vec{\nabla}\right)B_1 + \frac{1}{4\pi}\frac{B_1}{r}\frac{\partial \vec{B}_0}{\partial \phi}$$

$$\frac{\partial \hat{r}}{\partial \phi} = \hat{\phi}$$

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = \hat{\phi}\frac{1}{4\pi}\left(\vec{B}_0 \cdot \vec{\nabla}\right)B_1 + \frac{1}{4\pi}\frac{B_1 B_{0r}}{r}\hat{\phi}$$

$$\vec{v} = v\hat{\phi}$$

$$\rho_0 \frac{\partial v}{\partial t} = \frac{\left(\vec{B}_0 \cdot \vec{\nabla}\right)B_1}{4\pi} + \frac{B_1 B_{0r}}{4\pi r}$$

# Thin Tube Approximation

$$\rho_0 \frac{\partial v}{\partial t} = \frac{\left(\vec{B}_0 \cdot \vec{\nabla}\right) B_1}{4\pi} + \frac{B_1 B_{0r}}{4\pi r}$$ Azimuthal Momentum Equation

$$\frac{\partial B_1}{\partial t} = \left(\vec{B}_0 \cdot \vec{\nabla}\right) v - \frac{v B_{0r}}{r}$$ Azimuthal Induction Equation

Note we haven't used the fact that the tube is thin yet! Let's do it now. Use the solenoidal condition to eliminate $\boldsymbol{B_{0r}}$.

$$\vec{\nabla} \cdot \vec{B}_0 = \frac{1}{r} \frac{\partial}{\partial r}\left(r B_{0r}\right) + \frac{\partial B_{0z}}{\partial z} = 0$$

Integrate in radius and note that to lowest order $B_{0z} = B_0 \neq B_0(r)$

$$r B_{0r} + \frac{r^2}{2} \frac{\partial B_0}{\partial z} = 0 \longrightarrow B_{0r} = -\frac{r}{2} \frac{\partial B_0}{\partial z}$$

# Simplify the Equations

$$B_{0r} = -\frac{r}{2}\frac{\partial B_0}{\partial z}$$

Since the radial field component is small (because the tube is thin)

$$\vec{B}_0 \cdot \vec{\nabla} = B_{0r}\frac{\partial}{\partial r} + B_{0z}\frac{\partial}{\partial z} \qquad \approx B_0 \frac{\partial}{\partial z}$$

Use these expansions in the momentum and induction equations

$$\rho_0 \frac{\partial v}{\partial t} = \frac{\left(\vec{B}_0 \cdot \vec{\nabla}\right) B_1}{4\pi} + \frac{B_1 B_{0r}}{4\pi r} \longrightarrow \rho_0 \frac{\partial v}{\partial t} = \frac{B_0}{4\pi}\frac{\partial B_1}{\partial z} - \frac{B_1}{8\pi}\frac{\partial B_0}{\partial z}$$

$$\frac{\partial B_1}{\partial t} = \left(\vec{B}_0 \cdot \vec{\nabla}\right) v - \frac{v B_{0r}}{r} \longrightarrow \frac{\partial B_1}{\partial t} = B_0 \frac{\partial v}{\partial z} + \frac{v}{2}\frac{\partial B_0}{\partial z}$$

# Utilize the Scale Height

$$\frac{\partial}{\partial z}\left(\frac{B_0^2}{8\pi}\right) = -\frac{1}{H_P}\left(\frac{B_0^2}{8\pi}\right) \longrightarrow \boxed{\frac{\partial B_0}{\partial z} = -\frac{B_0}{2H_P}}$$

Insert into the momentum and induction equations

$$\rho_0 \frac{\partial v}{\partial t} = \frac{B_0}{4\pi}\frac{\partial B_1}{\partial z} - \frac{B_1}{8\pi}\frac{\partial B_0}{\partial z} \longrightarrow \boxed{\frac{\partial v}{\partial t} = \frac{B_0}{4\pi\rho_0}\left(\frac{\partial}{\partial z} + \frac{1}{4H_P}\right)B_1}$$

$$\frac{\partial B_1}{\partial t} = B_0\frac{\partial v}{\partial z} + \frac{v}{2}\frac{\partial B_0}{\partial z} \longrightarrow \boxed{\frac{\partial B_1}{\partial t} = B_0\left(\frac{\partial}{\partial z} - \frac{1}{4H_P}\right)v}$$

# Derive a Wave Equation

$$\frac{\partial v}{\partial t} = \frac{B_0}{4\pi\rho_0}\left(\frac{\partial}{\partial z} + \frac{1}{4H_P}\right)B_1$$

Azimuthal Momentum Equation

$$\frac{\partial B_1}{\partial t} = B_0\left(\frac{\partial}{\partial z} - \frac{1}{4H_P}\right)v$$

Azimuthal Induction Equation

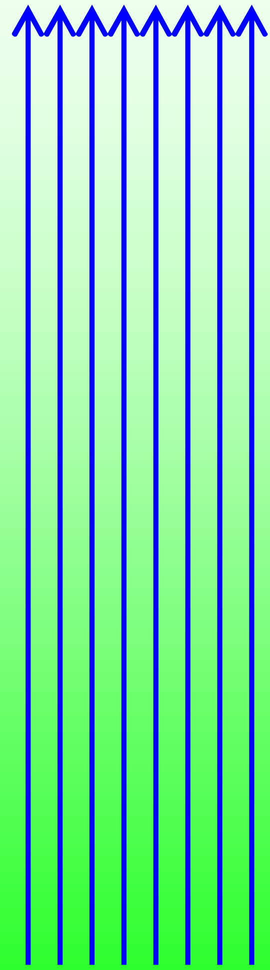
Take a temporal derivative of the momentum equation and insert the induction equation (after some algebra and calculus)

$$\frac{\partial^2 v}{\partial t^2} = V_A^2\left\{\frac{\partial^2}{\partial z^2} - \frac{1}{2H_P}\frac{\partial}{\partial z} + \frac{1}{16H_P^2}\left(1 + 4\frac{dH_P}{dz}\right)\right\}v$$

# No Gravity Limit

In the absence of gravity, the tube is straight sided Without flaring. The field strength is constant with height.

$$H_P^{-1} = 0$$



Therefore,

$$\frac{\partial^2 v}{\partial t^2} = V_A^2 \frac{\partial^2 v}{\partial z^2}$$

In this limit the torsional Alfvén wave propagates at the Alfvén speed without change in amplitude.

Different flux surfaces can oscillated independently.

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STFC summer school
September 2010

The University Of Sheffield.

# Standard Form

$$\left\{\frac{d^2}{dz^2} - \frac{1}{2H_P}\frac{\partial}{\partial z} + \frac{\omega^2}{V_A^2} + \frac{1}{16H_P^2}\left(1 + 4\frac{dH_P}{dz}\right)\right\} v = 0$$

We can transform it into standard form

$$v = f\psi$$

$$f\frac{d^2v}{dz^2} + \left[2\frac{df}{dz} - \frac{1}{2H_P}f\right]\frac{dv}{dz} + \left[\frac{\omega^2}{V_A^2} + \frac{1}{16H_P^2}\left(1 + 4\frac{dH_P}{dz}\right) + \frac{d^2f}{dz^2} - \frac{1}{2H_P}\frac{df}{dz}\right] v = 0$$

$$2\frac{df}{dz} - \frac{1}{2H_P}f = 0 \longrightarrow f = P_0^{-1/4}$$

$$\text{since } \frac{1}{P_0}\frac{dP_0}{dz} = -\frac{1}{H_P}$$

Miraculously, after substituting in all the terms

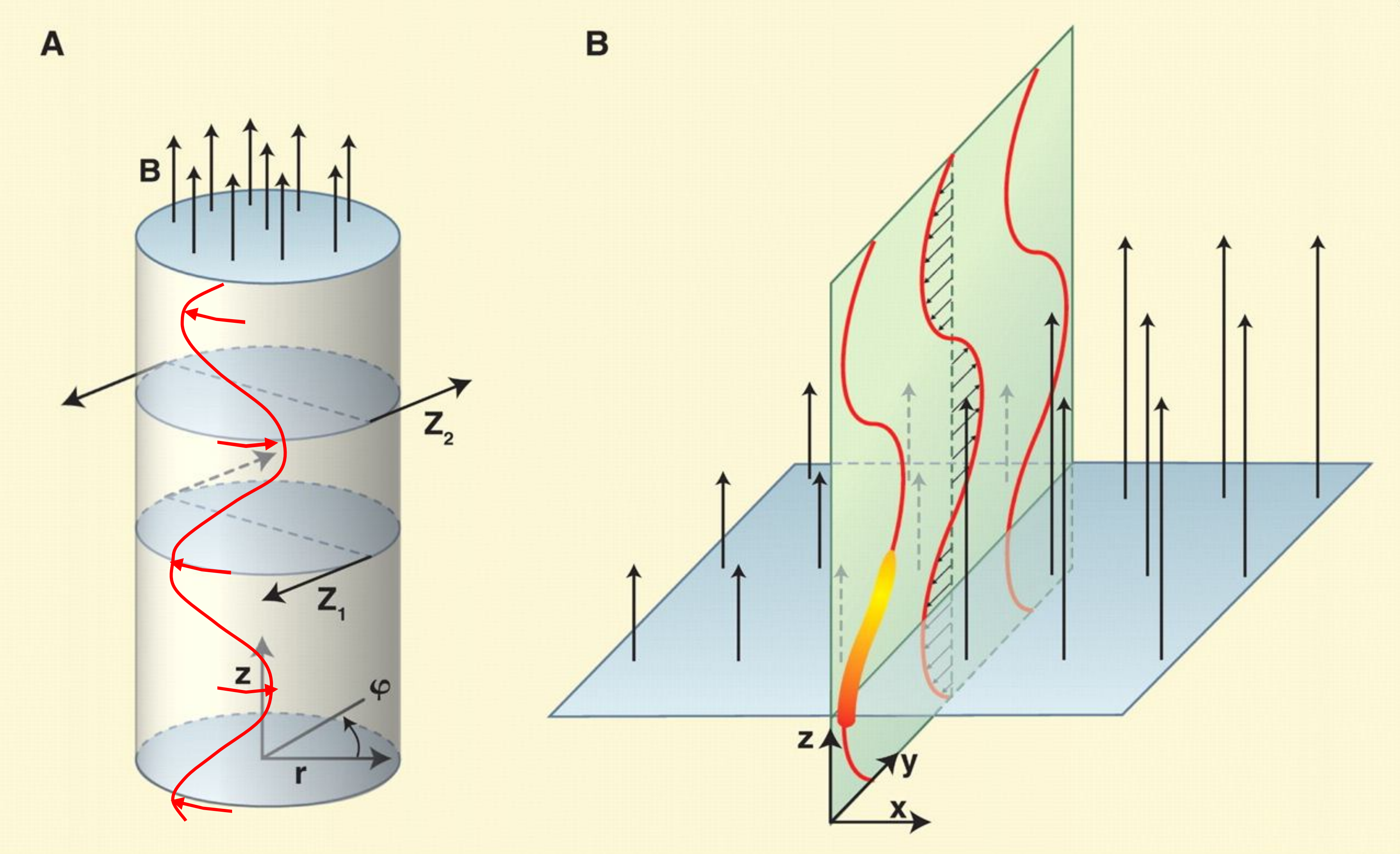
$$\boxed{\frac{d^2\psi}{dz^2} + \frac{\omega^2}{V_A^2}\psi = 0} \longleftarrow$$

There is no cutoff frequency!

# Torsional Alfvén Wave Properties

- Torsional (Azimuthal)
- Incompressive
- Field Following
- Tension Wave
- Propagates at the Alfvén speed
- Lacks a cutoff frequency

# Torsional and Shear Alfvén Waves

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Acknowledgements: Fedun+Erdelyi

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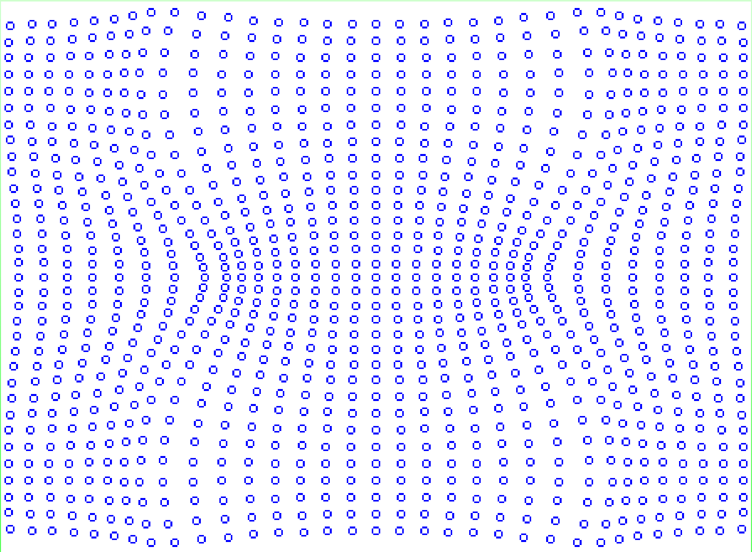
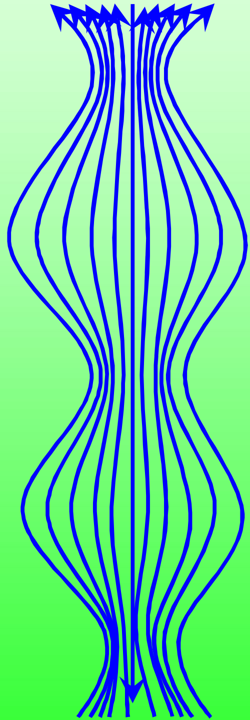
# Sausage Waves

axisymmetric pressure waves
(displacement parallel to the field)

$$m = 0$$



Graphics: N. Gareth (Univ. of Sheffield)



Sausage Mode
Magnetic axis does not move

$$\vec{v}(r, \phi, z, t) = v_{\parallel}(z, t)\hat{z}$$

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# Sausage Wave Equations

Start from a familiar place.

$$\frac{\partial \rho_1}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right)\rho_0 = -\rho_0 \vec{\nabla} \cdot \vec{v}$$

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla}\left(P_1 + \frac{\vec{B}_0 \cdot \vec{B}_1}{4\pi}\right) + \frac{\left(\vec{B}_0 \cdot \vec{\nabla}\right)\vec{B}_1 + \left(\vec{B}_1 \cdot \vec{\nabla}\right)\vec{B}_0}{4\pi} + \rho_1 \vec{g}$$

$$\frac{\partial P_1}{\partial t} + v_z \frac{dP_0}{dz} = -\rho_0 c_s^2 \vec{\nabla} \cdot \vec{v}$$

$$\frac{\partial \vec{B}_1}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right)\vec{B}_0 = \left(\vec{B}_0 \cdot \vec{\nabla}\right)\vec{v} - \vec{B}_0\left(\vec{\nabla} \cdot \vec{v}\right)$$

# Series Expansion in Radius

Since the tube is thin and axisymmetric we may seek a series expansion where the radius is treated as a small parameter.

$$\rho(r,z,t) = \rho(z,t) + r\rho^{(1)}(z,t) + \mathcal{O}(r^2)$$

$$P(r,z,t) = P(z,t) + rP^{(1)}(z,t) + \mathcal{O}(r^2)$$

$$v_z(r,z,t) = v_\parallel(z,t) + rv_\parallel^{(1)}(z,t) + \mathcal{O}(r^2)$$

$$v_r(r,z,t) = rv_r^{(1)}(z,t) + \mathcal{O}(r^2)$$

The axis can't move

$$B_z(r,z,t) = B_\parallel(z,t) + rB_\parallel^{(1)}(z,t) + \mathcal{O}(r^2)$$

$$B_r(r,z,t) = rB_r^{(1)}(z,t) + \mathcal{O}(r^2)$$

# Lowest Order Approximation

Insert these expansions into the linearized MHD equations and keep only the terms that are lowest order in radius.

$$\frac{\partial \rho_1}{\partial t} = -v_{\parallel} \frac{d\rho_0}{dz} - \rho_0 \Delta$$ Continuity

$$\rho_0 \frac{\partial v_{\parallel}}{\partial t} = -\frac{\partial P_1}{\partial z} - g\rho_1$$ Momentum

$$\frac{\partial P_1}{\partial t} = g\rho_0 v_{\parallel} - \rho_0 c_s^2 \Delta$$ Internal Energy

$$\frac{\partial B_{\parallel}}{\partial t} = -v_{\parallel} \frac{dB_0}{dz} - B_0 \Delta + B_0 \frac{\partial v_{\parallel}}{\partial z}$$ Induction

$$\Delta = \vec{\nabla} \cdot \vec{v} = \frac{\partial v_{\parallel}}{\partial z} + 2 v_r^{(1)}$$ Divergence

These equations only depend on height and time.

# Lateral Boundary Condition

This system has six variables and only five equations. We need another relation. We need another relation to close the system. We can get this relation by thinking about the conditions on the sides of the tube.

Since the tube is thin and can't support pressure gradients across it, the total pressure in the tube must be constant as a function of radius and equal the external value.

$v_r^{(1)}$

$$P_1 + \frac{B_0 B_\parallel}{4\pi} = \delta P_e = 0$$

Pressure Equilibration

If we use this equation instead of the equation for the divergence, we have five variables and five equations.

Note:
$\boldsymbol{\delta P_e}$ could be nonzero!
This permits external acoustic waves and convection to drive tube waves.

# Solve for the Velocity

Fourier transform the equations in time and eliminate all variables except the vertical velocity.

$$\left\{\frac{d^2}{dz^2} - \frac{1}{H_P}\frac{d}{dz} + \left[\frac{\omega^2 - N^2}{c_T^2} + \left(1 - \frac{\gamma}{2}\right)\frac{N^2}{c_s^2}\right]\right\} v_\parallel = 0$$

$$N^2 = \frac{g}{c_s^2}\left(\frac{dc_s^2}{dz} + (\gamma - 1)g\right)$$

Buoyancy Frequency

$$c_T^2 = \frac{c_s^2 V_A^2}{c_s^2 + V_A^2}$$

Tube Speed


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# No Gravity Limit

If there was no gravity, the tube would have straight sides and be invariant along its length.

$$N^2 = 0$$
$$H_p^{-1} = 0$$

$$\left\{ \frac{d^2}{dz^2} - \frac{1}{H_P}\frac{d}{dz} + \left[ \frac{\omega^2 - N^2}{c_T^2} + \left(1 - \frac{\gamma}{2}\right)\frac{N^2}{c_s^2} \right] \right\} v_\parallel = 0$$

$$\left\{ \frac{d^2}{dz^2} + \frac{\omega^2}{c_T^2} \right\} v_\parallel = 0$$

In this limit, the sausage wave propagates with tube speed without change in amplitude.

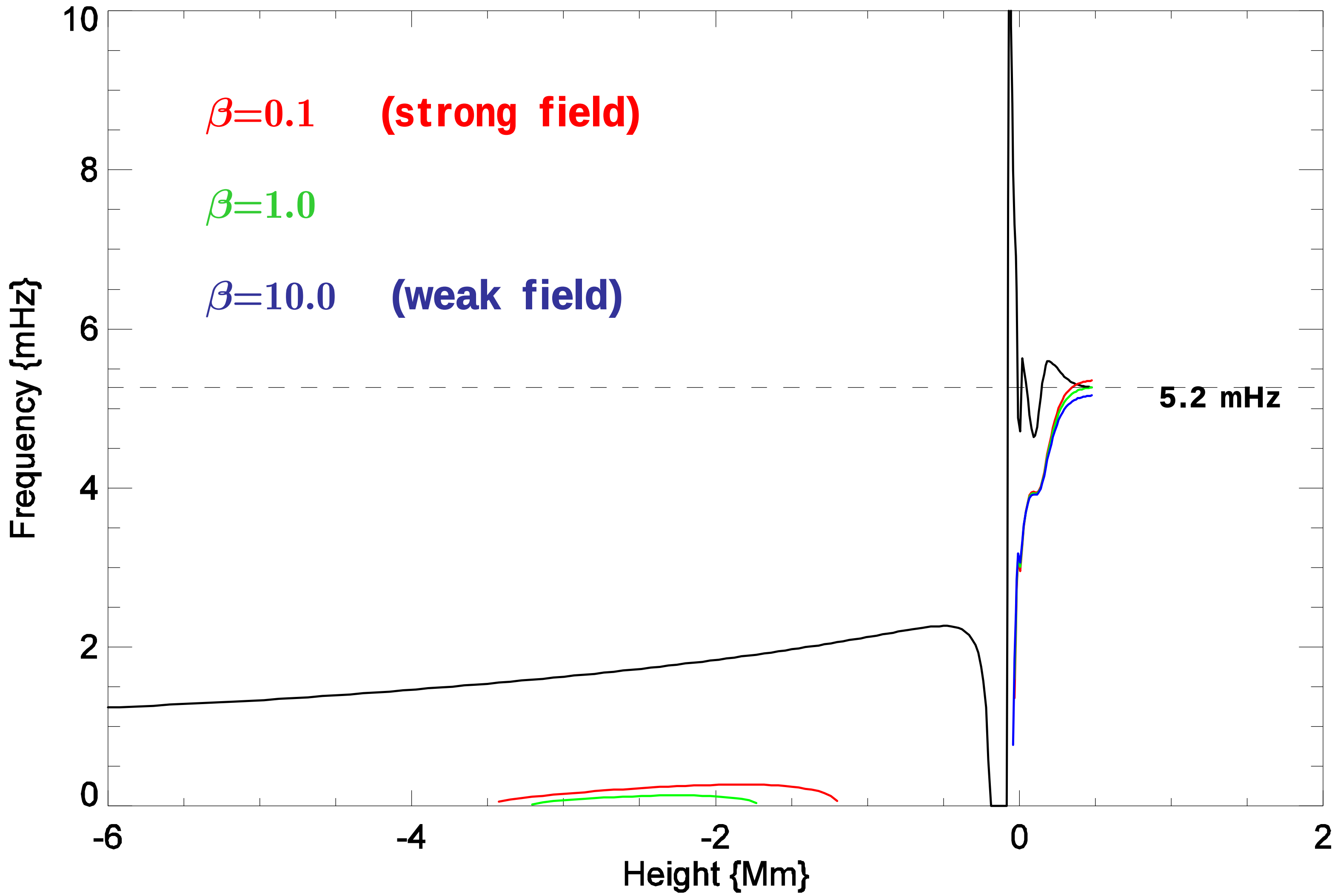
# Standard Form

After calculating all the coefficients, we can see that the sausage wave has a cutoff frequency

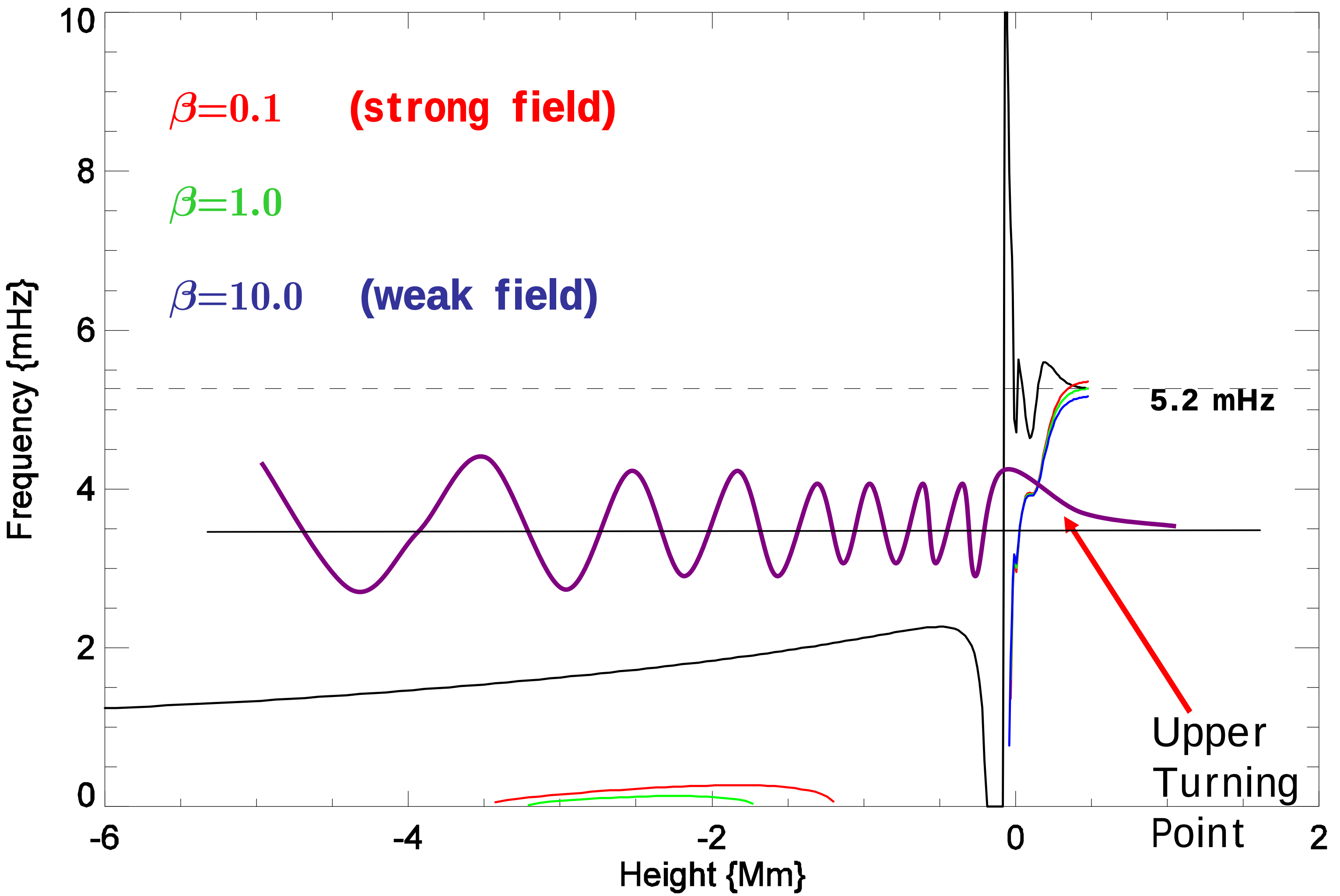
$$\frac{d^2v}{dz^2} + \left(\frac{\omega^2 - \omega_{\text{saus}}^2}{c_T^2}\right)v = 0$$

$$\omega_{\text{saus}}^2 = N^2 + g^2 \frac{c_T^2}{c_s^4}\left(\frac{3}{4}\gamma - 1\right)\left(\frac{3}{4}\gamma - 1 + \frac{1}{g}\frac{dc_s^2}{dz}\right)$$

# Cut-off Frequency



# Upper Turning Point

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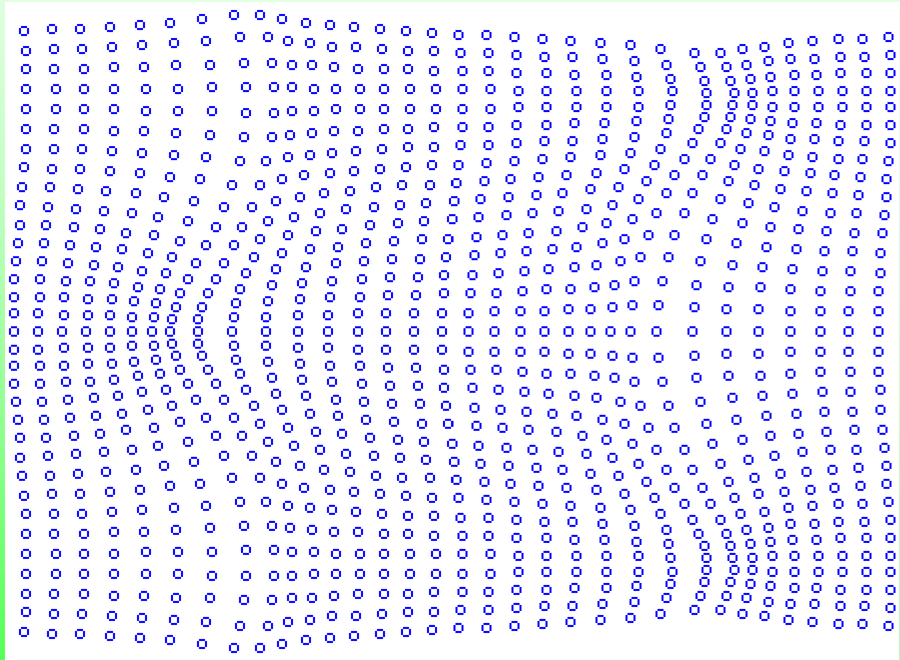
# Sausage Wave Properties

- Driven by gas pressure fluctuations along the tube
- The tube herniates to maintain constant total pressure
- Longitudinal
- Compressive
- Its technically a slow wave (slow tube wave), but it lacks many of the properties of a slow wave in a homogeneous media (i.e, when $V_A << c_s$ it is NOT tension driven).
- Propagates at the tube speed
- Possesses a cutoff frequency

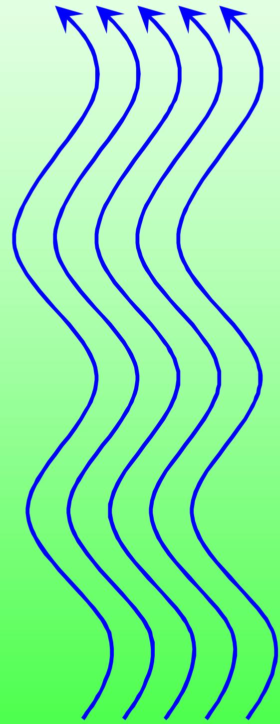
# Kink Tube Waves

$$\vec{v}(r,\varphi,z,t) = \sum_m \vec{u}(r,z)e^{im\varphi}e^{-i\omega t}$$

magnetic tension & buoyancy
(displacement perpendicular to the field)



Graphics: N. Gareth (Univ. of Sheffield)



$m = \pm 1$

Kink Mode
Magnetic axis moves

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# Kink Oscillations (Transverse Motions)

The inviscid ideal MHD momentum equation

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla}\left(P + \frac{B^2}{8\pi}\right) + \frac{\left(\vec{B}\cdot\vec{\nabla}\right)\vec{B}}{4\pi} + \rho\vec{g}$$

Total force per unit volume

Gas Pressure

Magnetic Pressure

Magnetic Tension

Gravity

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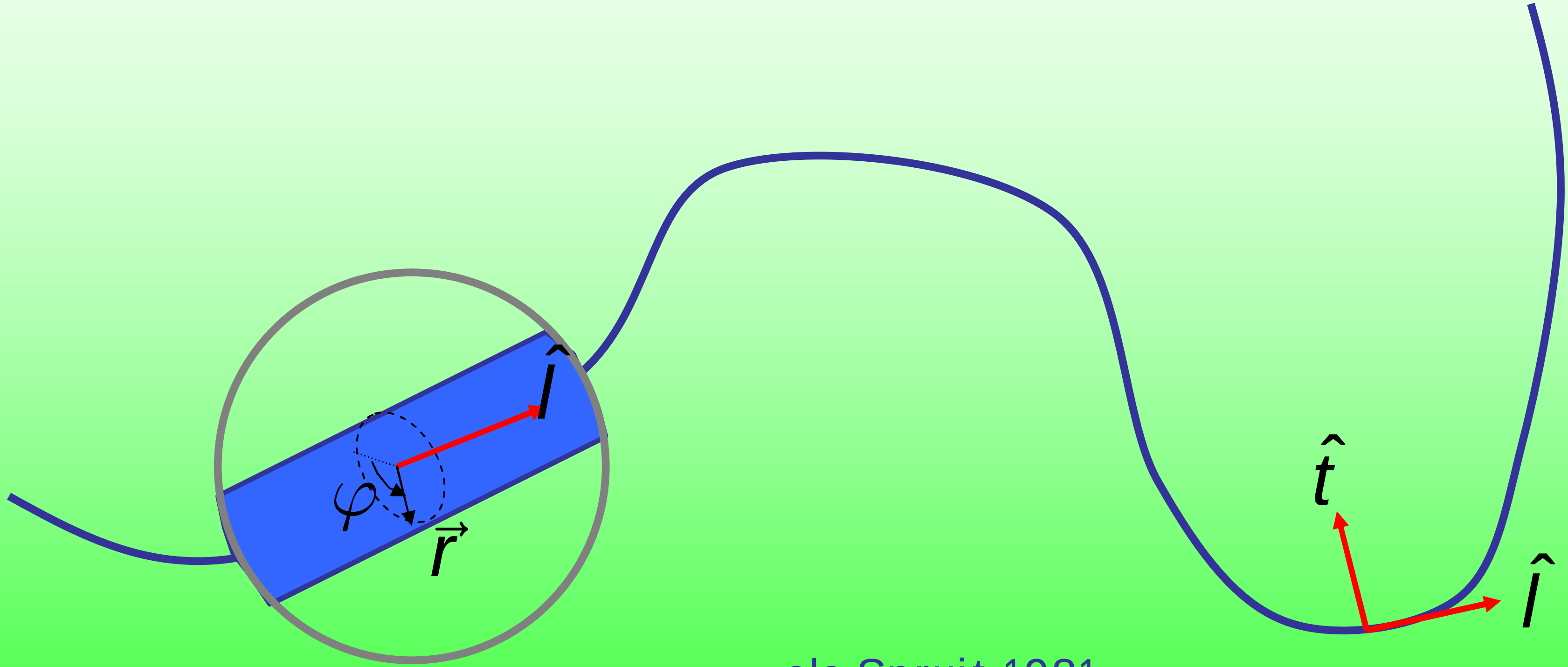
# Local Coordinate System




ala Spruit 1981

# Parallel and Perpendicular Components

Any vector quantity can be decomposed into a component that is parallel (‖) to flux tubes axis and a component perpendicular (⊥) to the axis.

$$\vec{V} = \vec{V}_{\parallel} + \vec{V}_{\perp}$$

$$\vec{V}_{\parallel} = \left(\hat{l} \cdot \vec{V}\right)\hat{l}$$

$$\vec{V}_{\perp} = \left(\hat{l} \times \vec{V}\right) \times \hat{l}$$

# Parallel and Perpendicular Forces

$$\rho \frac{D\vec{v}}{Dt} = -\vec{\nabla}\left(P + \frac{B^2}{8\pi}\right) + \frac{\left(\vec{B}\cdot\vec{\nabla}\right)\vec{B}}{4\pi} + \rho\vec{g}$$

The Lorentz force is transverse to the field lines. Therefore, only the gas pressure and gravity generate a parallel force component.

$$\vec{F}_{\parallel} = -\hat{l}\cdot\vec{\nabla}P + \rho\,\hat{l}\cdot\vec{g}$$

$$\vec{F}_{\perp} = -\hat{l}\times\vec{\nabla}\left(P + \frac{B^2}{8\pi}\right)\times\hat{l} + \frac{B^2}{4\pi}\hat{t} + \rho\left(\hat{l}\times\vec{g}\right)\times\hat{l}$$

$$\hat{t} = \left(\hat{l}\cdot\vec{\nabla}\right)\hat{l}$$ Curvature of the tube's path

# Kink Waves are transverse

Kink waves have transverse motions while sausage waves have longitudinal motions. Thus, let's concentrate our attention on the perpendicular force equation.

$$\vec{F}_{\perp} = -\hat{l} \times \vec{\nabla}\left(P + \frac{B^2}{8\pi}\right) \times \hat{l} + \frac{B^2}{4\pi}\hat{t} + \rho\left(\hat{l} \times \vec{g}\right) \times \hat{l}$$

This can be written a bit more compactly

$$\vec{F}_{\perp} = -\vec{\nabla}_{\perp}\left(P + \frac{B^2}{8\pi}\right) + \frac{B^2}{4\pi}\hat{t} + \rho\vec{g}_{\perp}$$

# Pressure Continuity

The total pressure must be continuous across the flux tube's interface

$$P + \frac{B^2}{8\pi} = P_e + \delta P_e$$

Total Pressure inside the tube

Unperturbed Pressure outside the tube

Back reaction pressure due to the motion of the tube

# Unperturbed Pressure

Outside the tube the Unperturbed pressure field obeys

$$\rho_e \frac{D\vec{v}_e}{Dt} = -\vec{\nabla} P_e + \rho_e \vec{g}$$

Therefore,

$$P + \frac{B^2}{8\pi} = P_e + \delta P_e$$

$$\vec{\nabla}\left(P + \frac{B^2}{8\pi}\right) = \rho_e \vec{g} - \rho_e \frac{D\vec{v}_e}{Dt} + \vec{\nabla}\delta P_e$$

# Substitute the external values

$$\vec{F}_{\perp} = -\vec{\nabla}_{\perp}\left(P + \frac{B^2}{8\pi}\right) + \frac{B^2}{4\pi}\hat{t} + \rho\vec{g}_{\perp}$$

$$\vec{\nabla}\left(P + \frac{B^2}{8\pi}\right) = \rho_e\vec{g} - \rho_e\frac{D\vec{v}_e}{Dt} + \vec{\nabla}\delta P_e$$

Combine these two equations

$$\vec{F}_{\perp} = \frac{B^2}{4\pi}\hat{t} + \left(\rho - \rho_e\right)\vec{g}_{\perp} + \rho_e\left(\frac{D\vec{v}_e}{Dt}\right)_{\perp} - \vec{\nabla}_{\perp}\delta P_e$$

Magnetic Tension | Buoyancy | External Force | Back Reaction

# Interpretation

Rewrite using the relative velocity

$$\vec{u}_{\text{rel}} = \vec{v}_{\perp} - \vec{v}_{\perp,\text{e}}$$

$$\vec{F} = -\rho_{\text{e}} \frac{d\vec{u}_{\text{rel}}}{dt} + \rho_{\text{e}} \left( \frac{D\vec{v}_{\perp,\text{e}}}{Dt} \right)_{\text{axis}} - \rho_{\text{e}} \left[ \left( \vec{u}_{\text{rel}} \cdot \vec{\nabla} \right) \vec{v}_{\perp,\text{e}} \right]_{\text{axis}} + 2\rho_{\text{e}} \left[ \vec{u}_{\text{rel}} \times \left( \nabla \times \vec{v}_{\perp,\text{e}} \right) \right]_{\text{axis}}$$

Enhanced Inertia (first term)

Unperturbed Force (second term)

Work required to maintain relative motion (third term)

Lift caused by flow around the cylinder (fourth term)

# Kink Equation

Force on a thin flux tube

$$\rho \frac{d\vec{v}_\perp}{dt} = \frac{B^2}{4\pi}\hat{t} + \left(\rho - \rho_e\right)\vec{g}_\perp + \rho_e \left(\frac{D\vec{v}_e}{Dt}\right)_\perp - \left\langle \vec{\nabla}_\perp \delta P_e \right\rangle$$

The last two terms are given by the 2D calculation

$$\vec{F} = -\rho_e \frac{d\vec{v}_\perp}{dt} + 2\rho_e \left(\frac{D\vec{v}_{\perp,e}}{Dt}\right)_{axis} - 2\rho_e \Omega_l \left[\left(\vec{v}_\perp - \vec{v}_{axis}\right) \times \hat{l}\right]$$

Giving

$$\rho \frac{d\vec{v}_\perp}{dt} = \frac{B^2}{4\pi}\hat{t} + \left(\rho - \rho_e\right)\vec{g}_\perp - \rho_e \frac{d\vec{v}_\perp}{dt} + 2\rho_e \left(\frac{D\vec{v}_{\perp,e}}{Dt}\right)_{axis} - 2\rho_e \Omega_l \left[\left(\vec{v}_\perp - \vec{v}_{axis}\right) \times \hat{l}\right]$$

# Final Form

Tube Inertia — Enhanced inertia — Magnetic Tension — Buoyancy — External Driving

$$\left(\rho + \rho_e\right)\frac{d\vec{v}_\perp}{dt} = \frac{B^2}{4\pi}\hat{t} + \left(\rho - \rho_e\right)\vec{g}_\perp + \vec{F}_{drive}$$

$$\vec{F}_{drive} = 2\rho_e \frac{D\vec{v}_{\perp,e}}{Dt} - 2\rho_e \Omega_l \left[\left(\vec{v}_\perp - \vec{v}_{\perp,e}\right) \times \hat{l}\right]$$

This is a general equation that describes the transverse motion of a thin flux tube. It can be used to describe

- kink waves
- the motion of a thin flux rising through convection
- etc.

# Kink Waves on a Vertical Tube

Differentiate wrt time, linearize and enforce the vertical axis of the tube

$$(\rho+\rho_e)\frac{\partial^2 v_\perp}{\partial t^2} = g(\rho-\rho_e)\frac{\partial v_\perp}{\partial z} + \frac{B^2}{4\pi}\frac{\partial^2 v_\perp}{\partial z^2}$$

$$\frac{\partial^2 v_\perp}{\partial t^2} = g\,\frac{\rho-\rho_e}{\rho+\rho_e}\frac{\partial v_\perp}{\partial z} + \frac{B^2}{4\pi(\rho+\rho_e)}\frac{\partial^2 v_\perp}{\partial z^2}$$

Fourier Transform and rearrange

$$\left\{c_K^2\frac{d^2}{dz^2} + g\,\frac{\rho-\rho_e}{\rho+\rho_e}\frac{d}{dz} + \omega^2\right\}v_\perp = 0$$

Kink speed

$$c_K^2 = \frac{B^2}{4\pi(\rho+\rho_e)}$$

# Set in Standard Form

$$\left\{ c_K^2 \frac{d^2}{dz^2} + g \frac{\rho - \rho_e}{\rho + \rho_e} \frac{d}{dz} + \omega^2 \right\} v_\perp = 0$$

Find the cut-off frequency by putting the wave equation in standard form

$$v_\perp(z) = f(z) v(z)$$

Chose ***f (z)*** such that the first derivative term vanishes

$$\frac{d^2 v}{dz^2} + \left( \frac{\omega^2 - \omega_{\mathrm{kink}}^2}{c_K^2} \right) v = 0$$

$$\omega_{\mathrm{kink}}^2 = \frac{g}{2(2\beta + 1)} \frac{1}{c_K^2} \left( \frac{g}{2(2\beta + 1)} + \frac{dc_K^2}{dz} \right)$$

# Kink Wave Properties

- Driven by magnetic tension and buoyancy
- Like a transverse Alfvén wave with drag
- Transverse
- Nearly incompressive
- It's technically a fast magnetoacoustic wave. (Note its slower than the Alfven wave because of the enhanced inertia)
- Propagates at the kink speed
- Possesses a cutoff frequency

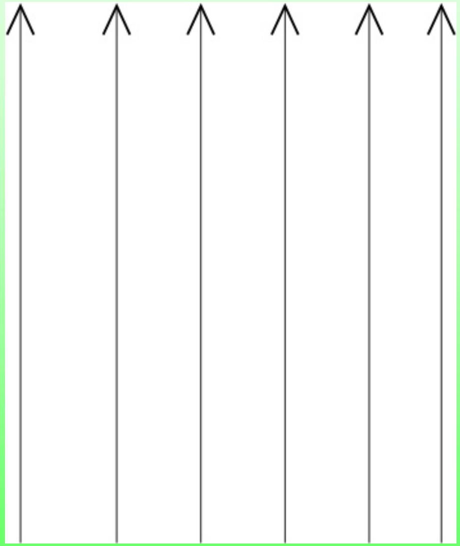
# Key speeds

- Sound speed $$c_s^2 = \frac{\gamma P_0}{\rho_0}$$
- Alfvén speed $$V_A^2 = \frac{B_0^2}{4\pi\rho_0}$$ Torsional Alfvén waves
- Tube speed $$c_T^2 = \frac{c_s^2 V_A^2}{c_s^2 + V_A^2}$$ Sausage waves
- Kink speed $$c_K^2 = \frac{B_0^2}{4\pi(\rho_0 + \rho_e)}$$ Kink waves

# No **g**, *non uniform* **B** and p

Seen: Alfvén, fast & slow speeds

- Now vary with x
- Leads to phase mixing
- Continuous spectra
- Resonant absorption
- Instabilities



$\mathbf{B}_0(x)$

$p_0(x)$

$\rho_0(x)$

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# Non-uniform, ideal MHD plasmas

Consider $\rho_0(x), \quad c_s^2(x), \quad c_A^2(x)$

Can derive linearised equation of motion

$$\frac{d}{dx}\left(\frac{N}{D}\frac{dv_x}{dx}\right) + \left(\omega^2 - k_\parallel^2 c_A^2\right)v_x = 0$$

The numerator and denominator are

$$N = \left(c_A^2(x) + c_s^2(x)\right)\left(\omega^2 - k_\parallel^2 c_A^2(x)\right)\left(\omega^2 - k_\parallel^2 c_A^2(x) c_s^2(x)/(c_A^2(x) + c_s^2(x))\right)$$

$$D = \left(\omega^2 - \omega_I^2(x)\right)\left(\omega^2 - \omega_{II}^2(x)\right)$$

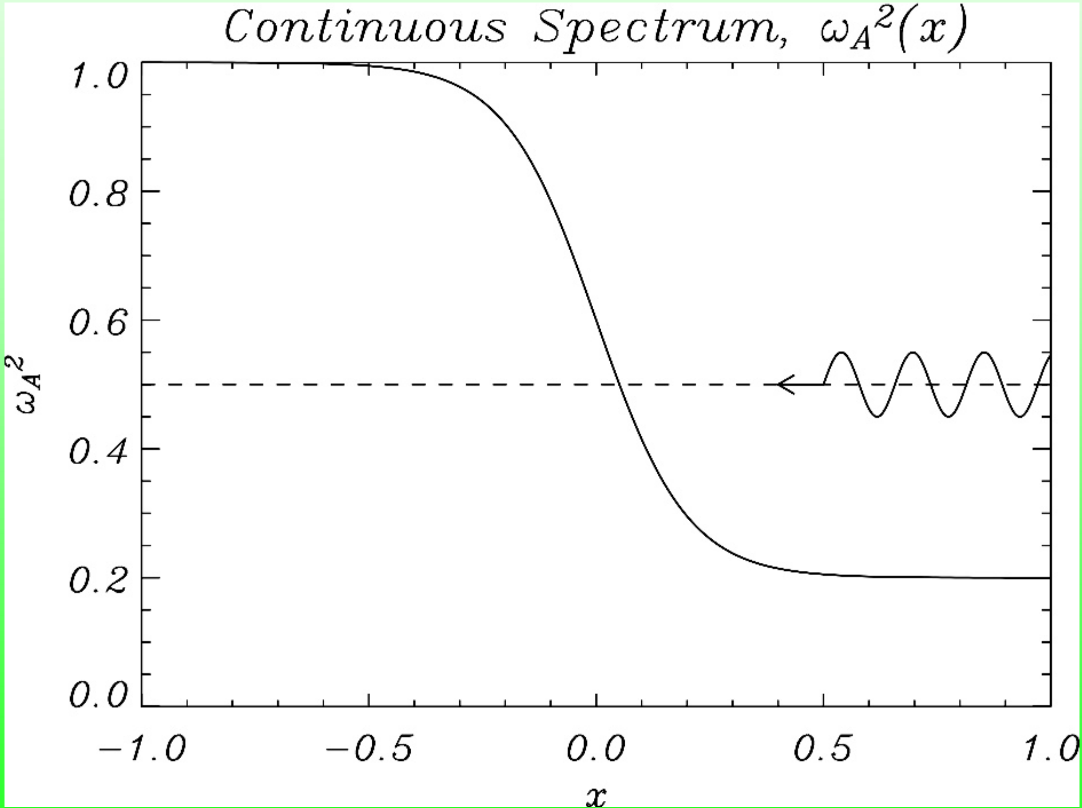
Zeros of N give Continuous Spectra (Alfvén, Slow)
Zeros of D give Turning Points

Parallel wavenumber $K_{II} = k_z$

Note continuous spectrum reaches zero if $k_z=0$ (no bending of $B_0$).

# Non-uniform plasmas

1. Continuous spectra allows Resonant Absorption
2. Incoming wave hits a Resonance Layer
3. Ideal MHD => Singularity
4. Non-ideal MHD => Damping




Fast wave, ($v_x$) coming in from right, hits resonance at $x_0$.

# solitons

Animation courtesy of
Dr. Dan Russell, Kettering University

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# MHD instabilities

Two methods: The Normal mode method and The energy principle

## The energy principle

Perturb the system by a small amount
Linearise the MHD equations,
Calculate the P.E. of the system
If P.E. (perturbed) > P.E. (unperturbed)
The system is stable against this perturbation
Generally used when there are complex geometric configurations.

## Linearised Ideal MHD Equations: Normal Modes

$$\rho_0 \frac{\partial^2 \mathbf{v}_1}{\partial t^2} = \mathbf{F}(\mathbf{v}_1),$$

Since $t$ does not appear in the coefficients, we may assume $\mathbf{v}_1 = \mathbf{v}_1 e^{i\omega t}$ and we have the eigenvalue problem

$$-\omega^2 \rho_0 \mathbf{v}_1 = \mathbf{F}(\mathbf{v}_1),$$

In ideal MHD $\omega^2$ is real. Therefore in ideal MHD only have waves (if all $\omega^2 > 0$) or instabilities (if one $\omega^2 = -\sigma^2 < 0$ with growth rate $\sigma$).
We find the *normal modes* of the equilibrium.

A normal mode of an oscillating system is a pattern of motion in which all parts of the system move sinusoidally with the same frequency & in phase.

# Rayleigh -Taylor instability (gravity and horizontal $B$)

Simplest instability to understand is *Rayleigh-Taylor* instability. Can occur in solar prominences (dense plasma lies above lighter plasma).
Consider two fluids separated by an interface at $z = 0$ with $\rho^+$ and $\rho^-$ (and no $B$ initially). Hence,

$$\rho = \begin{cases} \rho^+ & z > 0, \\ \rho^- & z < 0. \end{cases}$$

Pressure satisfies

$$\frac{dp}{dz} = -\rho g,$$

in each fluid.For equilibrium the *pressure must be continuous at the interface.*

- Disturbances of form $f(z)e^{i(kx-\omega t)}$.
- Solve linearised equations in each fluid with $v_z \to 0$ as $z \to \pm\infty$.
- Assume that the perturbations are *incompressible* $\nabla \cdot \mathbf{v} = 0$.
- The dispersion relation from continuity of vertical velocity and pressure at the interface (remembering interface is actually moved by the perturbation).

$$\omega^2 = kg\frac{\rho^- - \rho^+}{\rho^- + \rho^+}.$$

The plasma is unstable if $\rho^+ > \rho^-$ so that denser fluid is on top of lighter fluid.

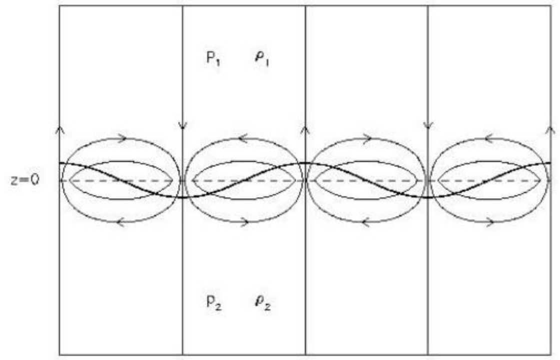



Figure: Rayleigh Taylor instability.

The form of perturbation of interface is shown in Figure 1 along with the velocity streamlines. Most unstable mode for $k \to \infty$. Hence, very short wavelengths.

- Add uniform magnetic field at angle $\theta$ to $x$ axis.
- As above, get dispersion relation

$$\omega^2 = kg\frac{\rho^- - \rho^+}{\rho^- + \rho^+} + \frac{2}{\mu}\frac{(kB\cos\theta)^2}{\rho^- + \rho^+}.$$

- Extra term positive and stabilising. An Alfvén wave due to stabilising effect of magnetic tension.

## Example: prominence

The University Of Sheffield.

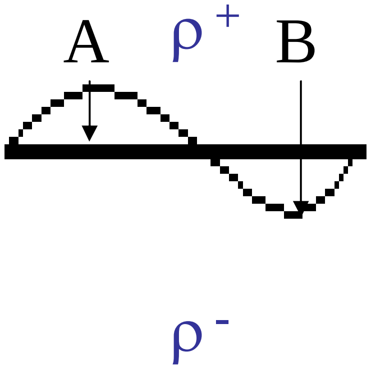
# Rayleigh-Taylor Instability
# Energy approach

Area A = area B

Centre of mass at +d/2 and –d/2

Upper fluid loses P.E. $-\rho^{+} gAd$

Lower fluid gains P.E. $+\rho^{-} gAd$



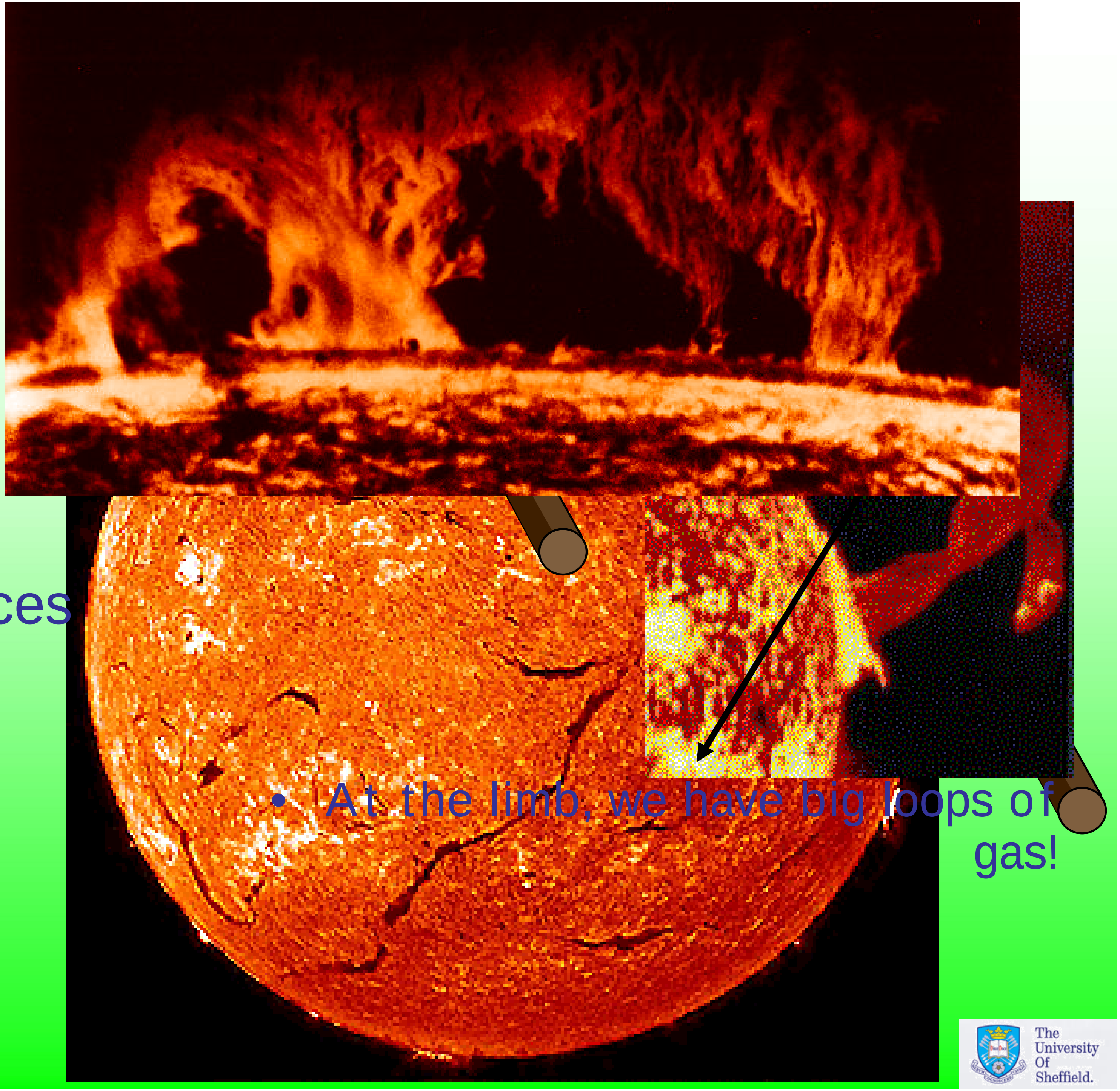

Change in P.E., $\delta W = \left(\rho^{-} - \rho^{+}\right) gAd$

$\rho^{-} > \rho^{+} \Rightarrow \delta W > 0$ STABLE

$\rho^{-} < \rho^{+} \Rightarrow \delta W < 0$ UNSTABLE

Example: prominence

- These are
- prominences



- At the limb, we have big loops of gas!

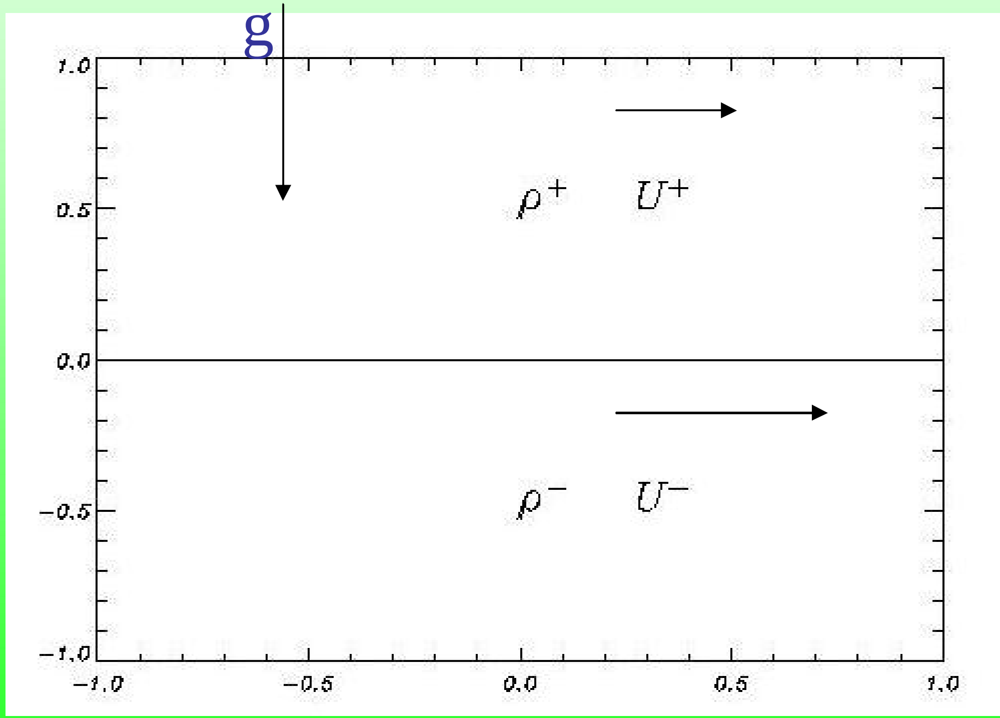
U(z), g, B

# Kelvin-Helmholtz (shear flow & gravity)

- Instability when horizontal shear flow **v**=(U(z),0,0) and density stratification due to gravity.
- Consider interface

$$\rho = \begin{cases} \rho^{+} & z > 0, \\ \rho^{-} & z < 0, \end{cases}$$

$$U = \begin{cases} U^{+} & z > 0, \\ U^{-} & z < 0, \end{cases}$$



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# Kelvin-Helmholtz (shear flow & gravity)

Assume incompressible, linearised MHD equations.

Assume $v_z = U^{\pm} e^{i(kx-\omega t)} e^{\mp kz}$

Use continuity of normal displacement and pressure. Dispersion relation is

$$\rho^+(\omega - kU^+)^2 + \rho^-(\omega - kU^-)^2 = kg(\rho^- - \rho^+)$$

$$\Rightarrow (\rho^+ + \rho^-)\omega^2 - 2k(\rho^+U^+ + \rho^-U^-)\omega + (\rho^+U^{+2} + \rho^-U^{-2}) - kg(\rho^- - \rho^+) = 0$$

Roots are real or complex conjugate pair $\omega = \Omega \pm i\sigma$

Unstable if $(U^- - U^+)^2 > \frac{g}{k}\frac{((\rho^-)^2 - (\rho^+)^2)}{\rho^+\rho^-}$

# Kelvin-Helmholtz (shear flow, mag. field & gravity)

Occurs for an initial shear flow and density stratification due to gravity, e.g. in helmet streamers and on the boundary between the fast and slow solar wind streams.
The initial shear flow is given by

$$\mathbf{v} = (U(z), 0, 0)$$

and the vertical component of the steady $(\partial/\partial t = 0)$ equation of motion

$$\frac{dp}{dz} + \frac{d}{dz}\left(\frac{B^2(z)}{2\mu}\right) = -\rho g.$$

The simplest example is to consider two uniform plasmas separated by an interface at $z = 0$, namely

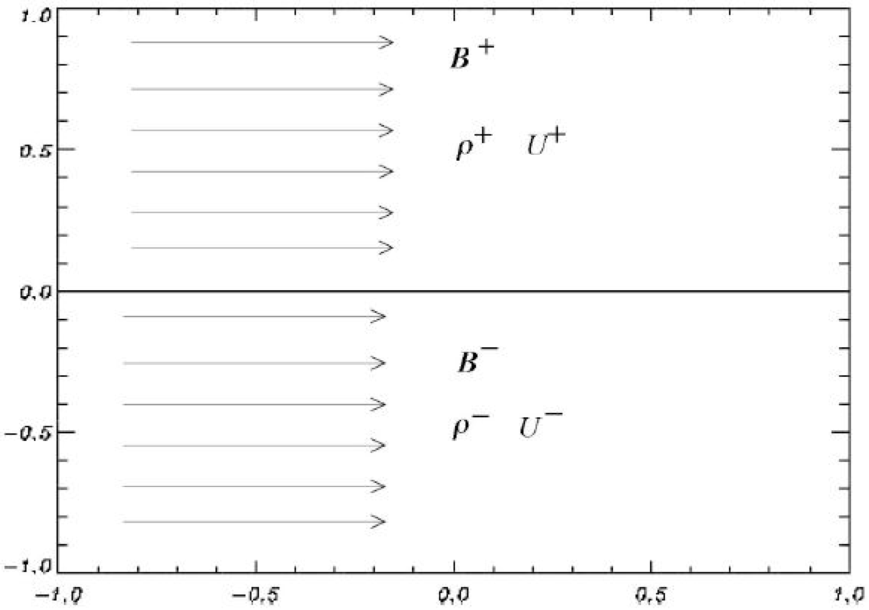



Figure: The initial state for the KelvinHelmholtz instability with a horizontal magnetic field.

The initial state consists of

$$\rho = \begin{cases} \rho^+ & z > 0, \\ \rho^- & z < 0, \end{cases}$$

$$U = \begin{cases} U^+ & z > 0, \\ U^- & z < 0. \end{cases}$$

$$B_x = \begin{cases} B^+ & z > 0, \\ B^- & z < 0. \end{cases}$$

In addition, need *total pressure is continuous* across interface. Assume perturbations are incompressible.Perturbations bend field lines and tension helps stabilise.Kelvin-Helmholtz instability can be stabilised for all $k$ if

$$\frac{B^{-2} + B^{+2}}{\mu \rho^- \rho^+} (\rho^- + \rho^+) \geq (U^- - U^+)^2 .$$

# Magnetic tension helps to stabilise K-H instability

Examples: Surfaces of the cometary tails
The edges of galactic and extragalactic jets

# Buoyancy and magnetic buoyancy instability

Buoyancy of a magnetic field is the reason large flux tubes rise to photosphere.Sunspots from as tubes break through the photosphere and emerge.

Consider a stratified, unmagnetised atmosphere, satisfying

$$\begin{aligned} \frac{dp}{dz} &= -\rho g, \\ p &= \frac{\rho R T}{\tilde{\mu}}, \\ T &= T_0(1 - mz). \end{aligned}$$

The linear temperature profile unstable to buoyancy instability. Use simple physical argument for instability conditions.

Consider a small fluid element. Imagine that this fluid element is lifted up a distance $dz$.

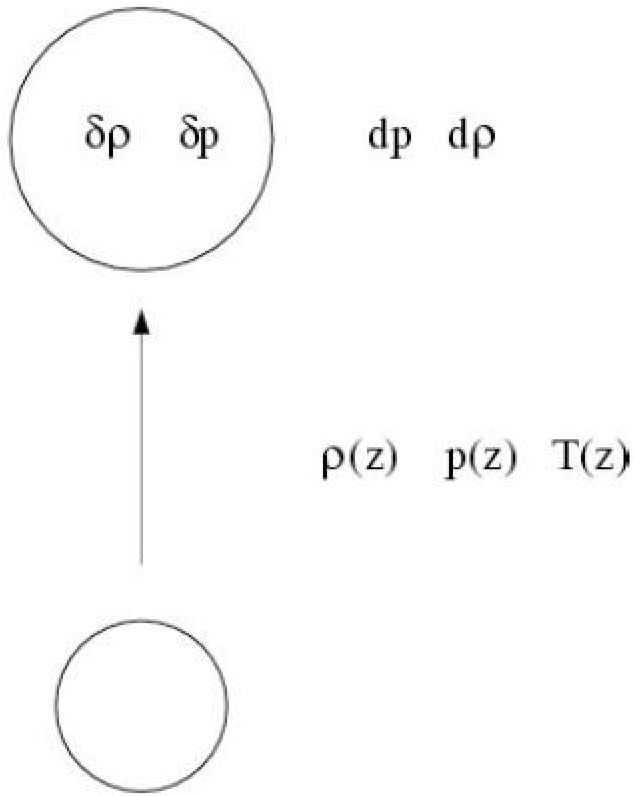



Figure: Physical explanation of the buoyancy instability.

- Surrounding plasma has pressure change of $dp$,

$$dp = \frac{dp}{dz}dz = -\rho g dz,$$

- density change of $d\rho = \frac{d\rho}{dz}dz$.
- Fluid element initially has higher pressure than the surrounding plasma and so expands until $\delta p = dp$.
- Ideal MHD has adiabatic energy form, $p/\rho^{\gamma} =$ constant.
- Thus, $\delta p = \frac{\gamma p}{\rho}\delta\rho = c_s^2\delta\rho = dp$.
- $$\delta\rho = \frac{\delta p}{c_s^2} = \frac{dp}{c_s^2} = \frac{dp/dz}{c_s^2}dz = -\frac{\rho g}{c_s^2}dz.$$
- Instability if $\delta\rho < d\rho = \frac{d\rho}{dz}dz$, or $-\frac{\rho g}{c_s^2} < \frac{d\rho}{dz}$.

## Brünt-Vaïsïla frequency

Defining *density scale height* $H = -\frac{\rho}{d\rho/dz}$ and the frequency, $N$, as

$$N^2 = \left(\frac{1}{H} - \frac{g}{c_s^2}\right),$$

- Buoyancy instability if

$$N^2 < 0.$$

- Another equivalent form for the Brünt-Vaïsïla frequency is

$$N^2 = \frac{g}{\gamma}\frac{d}{dz}\left[\log\left(\frac{p}{\rho^\gamma}\right)\right].$$

- Instability occurs if the entropy decreases with height.
- Can be rewritten in terms of the temperature gradient and instability occurs if

$$\frac{dT}{dz} < -\frac{\gamma - 1}{\gamma}\frac{\tilde{\mu} g}{R} = \left(\frac{dT}{dz}\right)_{ad}.$$

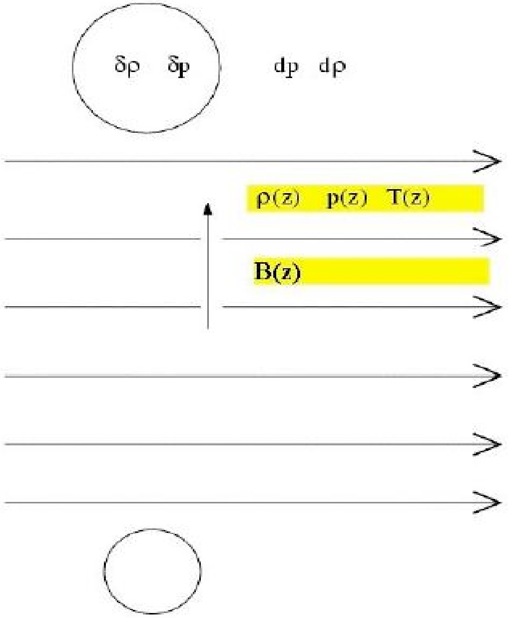



Figure: Physical explanation for the magnetic buoyancy instability.

- Fluid element expands adiabatically

$$\delta p = c_s^2 \delta\rho.$$

- Magnetic field evolves with $B/\rho$ held constant

$$\frac{\delta B}{B} = \frac{\delta\rho}{\rho}.$$

- Fluid element expands until in total pressure balance,

$$\delta p + \frac{B}{\mu}\delta B = dp + \frac{B}{\mu} dB.$$

Using our expressions for $\delta p$ and $\delta B$ in terms of $\delta\rho$, we have

$$\left(c_s^2 + c_A^2\right)\delta\rho = dp + \frac{B}{\mu} dB < (c_s^2 + c_A^2) d\rho.$$

- Include a horizontal magnetic field $(B(z), 0, 0)$ as well as $p(z)$ and $\rho(z)$.
- Assume a fluid element rises a distance $dz$.
- External atmosphere changes by $dp$, $BdB/\mu$ and $d\rho$.
- Fluid element changes by $\delta p$, $B\delta B/\mu$ and $\delta\rho$.
- There will be an instability if $\delta\rho < d\rho$.

- Thus, for instability

$$\frac{dp}{dz} + \frac{B}{\mu}\frac{dB}{dz} < \left(c_s^2 + c_A^2\right)\frac{d\rho}{dz}.$$

- This can be rewritten as

$$g c_A^2 \frac{d}{dz}\left[\log\left(\frac{B}{\rho}\right)\right] + c_s^2 N^2 < 0.$$

The plasma can be unstable with a magnetic field even when the square of the Brunt-Vaisila frequency is positive.

# Onset of Convection (temp gradient + gravity + dissipation but no B)

- Convection is set when plasma is heated from below.
- Include viscosity and thermal conduction, (missing from above).
- Use the Boussinesq approximation to filter out the sound waves; neglect density & temp. variations except in the gravity term.
- The density is expressed as $\rho = \rho_0(1 + \alpha T_1)$,

where T1 is the perturbed temperature and $\alpha$ is the volume expansion.

- The key parameter is the *Rayleigh number*, $Ra$,

$$Ra = \frac{g\alpha\Delta T}{\kappa\nu},$$

where $\Delta T$ is the temperature difference between the base of the convection zone and the photosphere, $\kappa$ is the thermal conductivity and $\nu$ is the viscosity.

- Convection when the Rayleigh number exceeds a critical value

$$Ra > Ra^* = \frac{27\pi^4}{4}.$$

This occurs for a particular wavenumber.

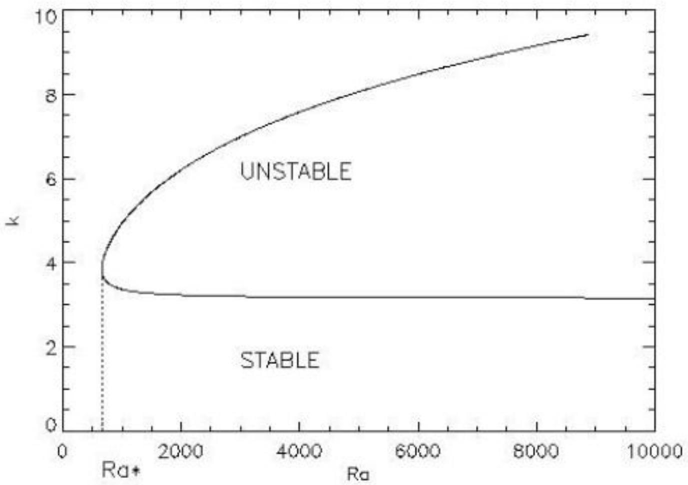



Figure: The marginal stability curve as a function of the wavenumber, $k$, and the Rayleigh number, $Ra$.

# Ideal MHD Kink & Sausage mode Instabilities

Consider equilibrium coronal loop modelled by a twisted cylinder.

$$\mathbf{B}_0 = (0, B_\theta(r), B_z(r)), \; p_0 = p(r),$$

and

$$B_z \frac{dB_z}{dr} + \frac{B_\theta}{r}\frac{d}{dr}(rB_\theta) + \mu\frac{dp}{dr} = 0.$$

Fourier analyse in $\theta$ and $z$ and assume, e.g.

$$v_r = v_r(r)e^{i(m\theta + kz - \omega t)}.$$

$m$ is the integer azimuthal wavenumber.

- The value of $m$ determines the mode.
- $m = 0$ is called the *sausage* mode,
- $m = 1$ the *kink* mode
- $m \geq 2$ *interchange* modes.

In the solar corona, $\beta \ll 1$, so that the gas pressure is often negligible. In this case, the first mode to become unstable is either the kink mode ($m = 1$) or the sausage mode ($m = 0$).

# Sausage mode instability

- Consider the sausage mode (m = 0) with an axialwavenumber k.
- Strong magnetic pressure at the compression and a weak magnetic pressure at the expansion.
- Pressure difference that squeezes the plasma from the compression into the expansion.
- Reducing p at the compression, there is nothing to stop the inward Lorentz force.
- Can be stabilised by adding an axial field to the equilibrium.

Compression and expansion produce magnetic tension & pressure forces.

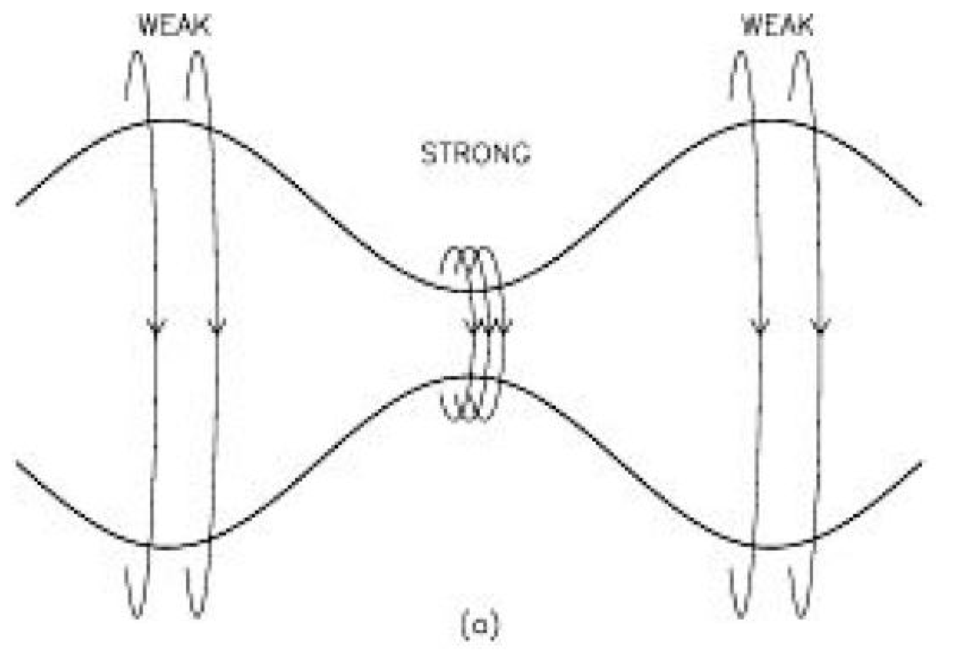
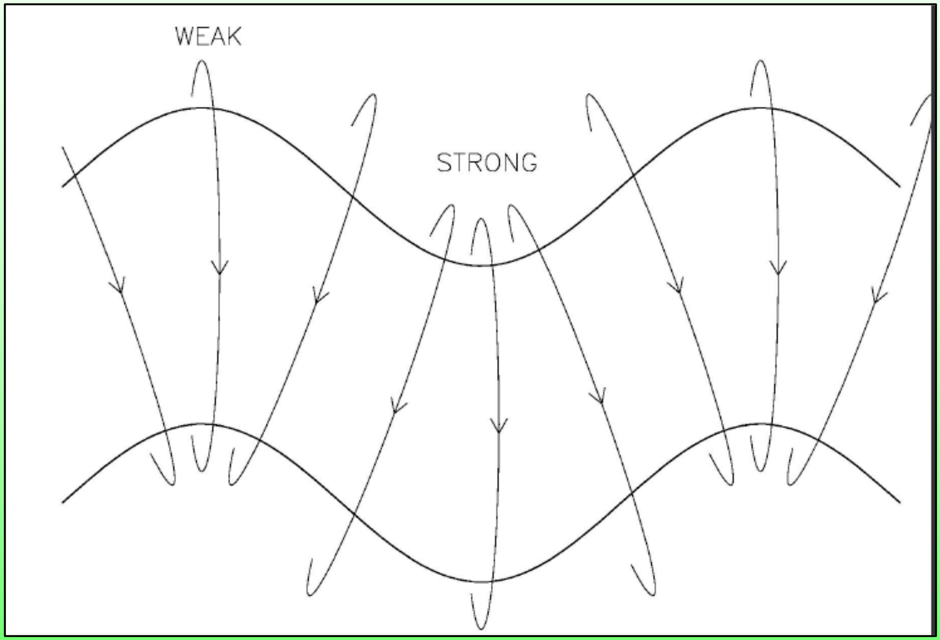



Figure: $m = 0$, sausage mode instability

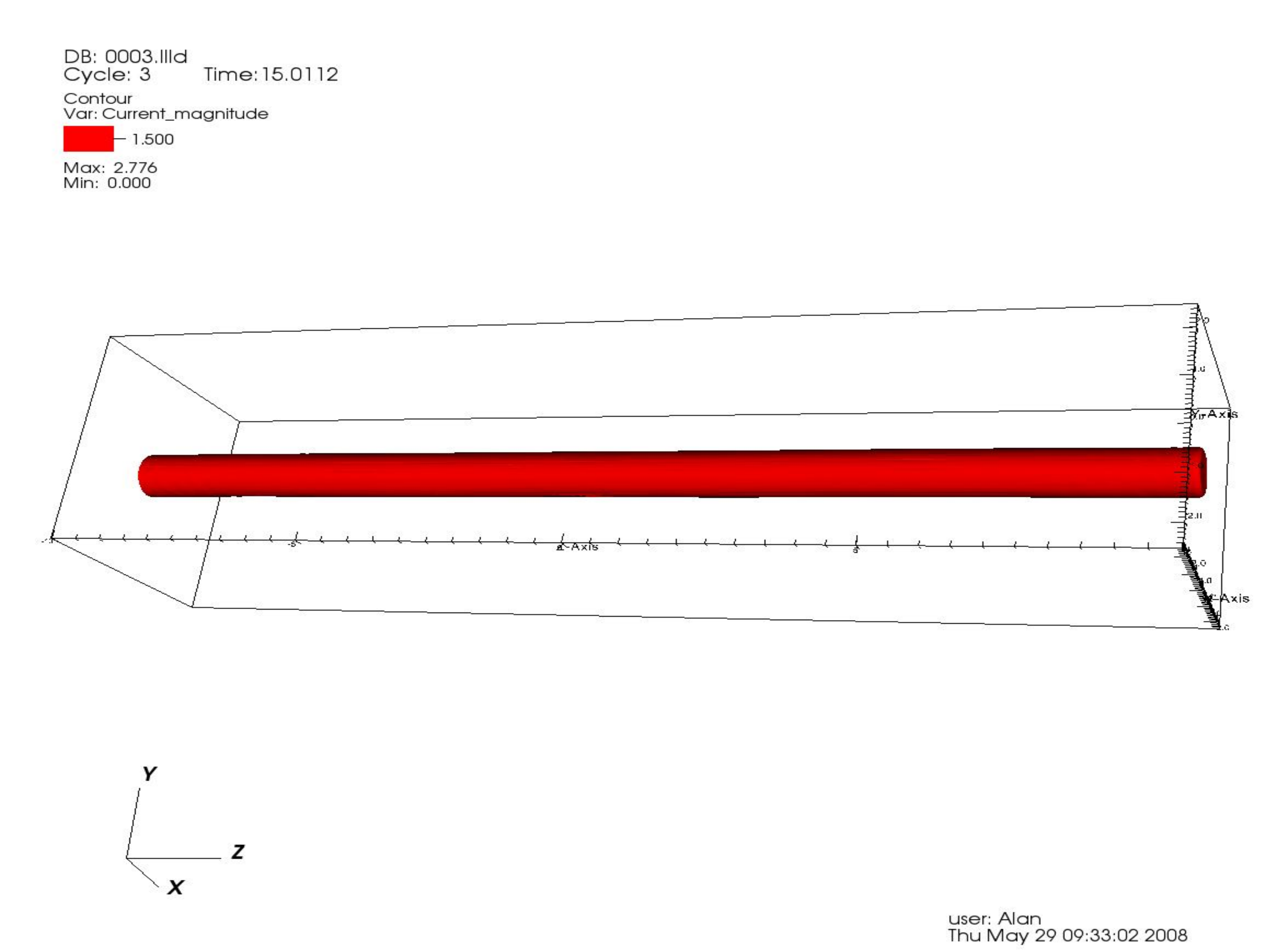
Examples: cometary tails, astrophysical sites showing filamentary structures.

# Kink mode instability

- Azimuthal field lines are brought close together on the inside of the bend, magnetic pressure will increase here.
- On the outside of bend the field lines are further apart, magnetic pressure is weaker.
- Hence, there is a magnetic pressure difference.




This lateral kink instability can be stabilised by including an axial magnetic field (adds tension)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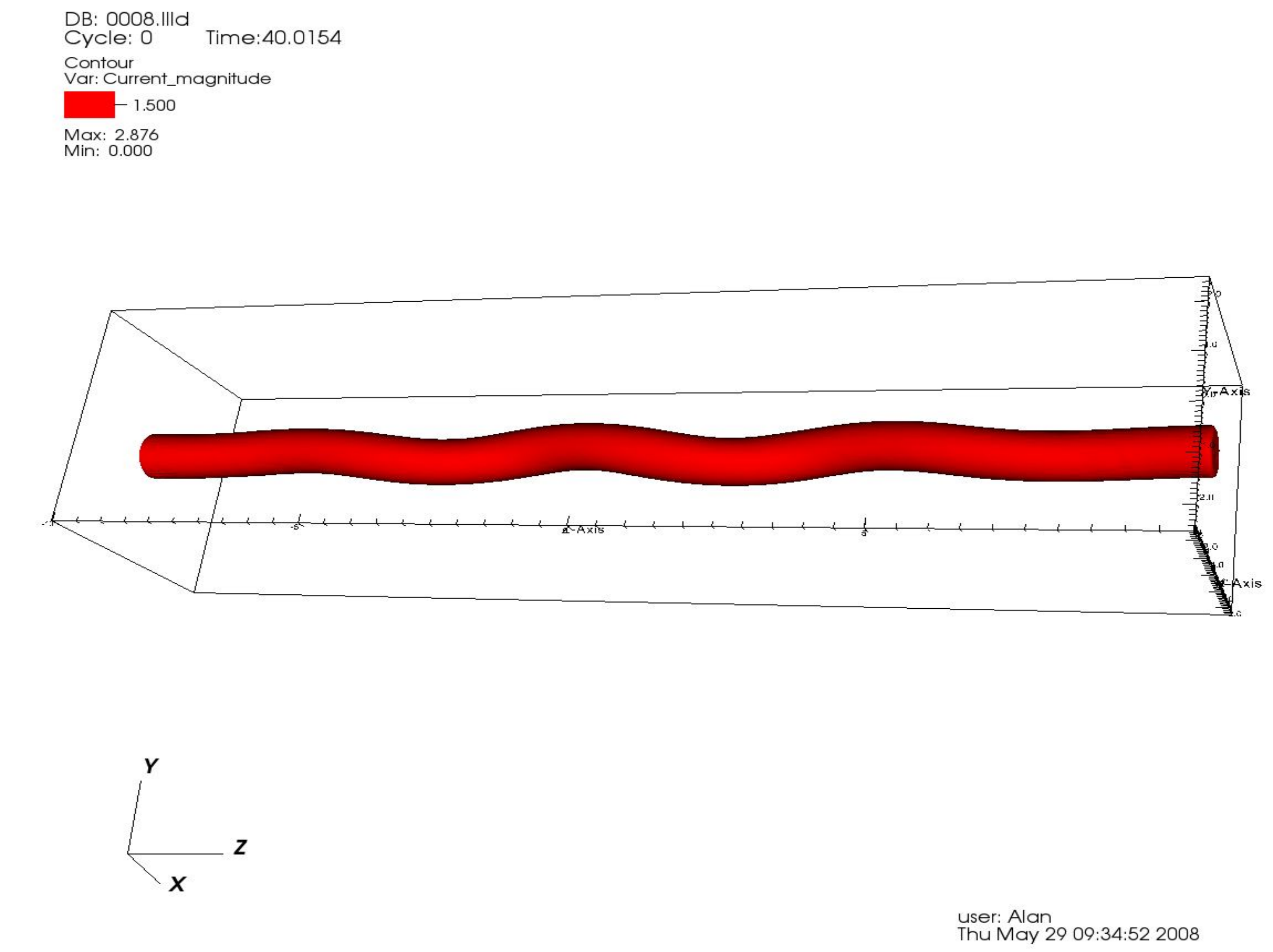
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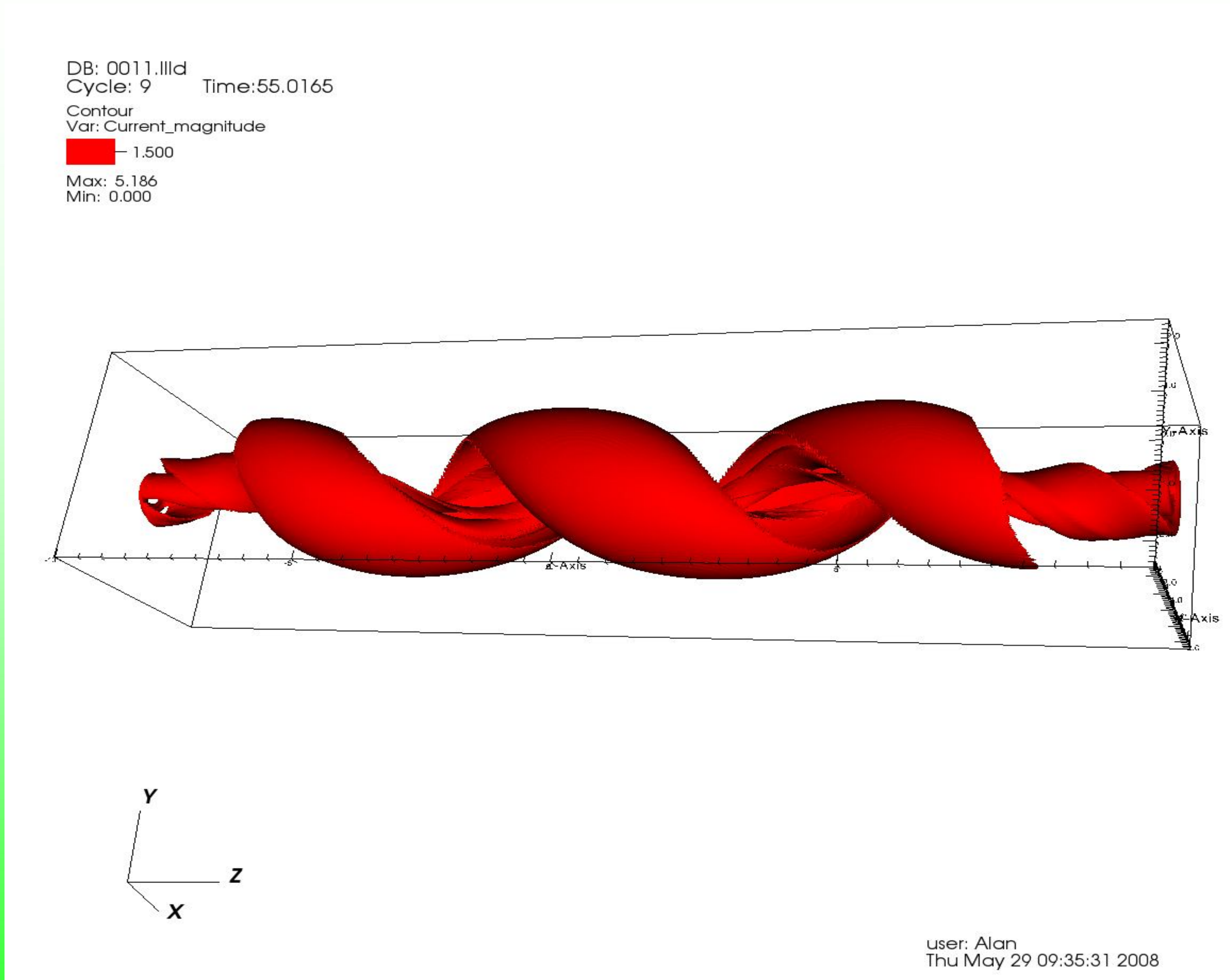




The University Of Sheffield.


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# Tearing Mode instability (resistivity & magnetic field)

We illustrate the effect of resistivity by considering the tearing mode instability in Cartesian geometry.

The tearing mode is a linear instability.

It is the only instability that allows a change in fieldline connectivity to occur.

Magnetic reconnection can result from the nonlinear development of the tearing mode.

## Equilibrium

The equilibrium is a sheared magnetic field

$$\mathbf{B}_0 = (0, B_{0y}(x), B_{0z}(x)),$$

with

$$\frac{d}{dx}\left(\frac{B_{0y}^2}{2} + \frac{B_{0z}^2}{2}\right) = 0 \quad \Rightarrow \quad B_{0y}^2 + B_{0z}^2 = \text{ constant.}$$

- Magnetic pressure is constant. No magnetic tension force.
- Assume $B_{0z}(0) = 0$.
- Stable to ideal MHD perturbations.
- Consider perturbations $f(x)e^{i(ly+kz)}e^{\sigma t}$.
- The resonant layer discussed above will form at

$$lB_{0y} + kB_{0z} = 0.$$

Consider ι = 0, the resonant layer will be at x = 0.

- The tearing mode due to small non-zero resistivity.
- Only important near $x = 0$ in a *boundary layer*.
- Outside boundary layer, resistivity is neglected.
- Determine a solution to the ideal MHD equations, a resistive MHD solution valid in the boundary layer and match the two solutions together.
- The matching process is only possible for particular values of the growth rate $\sigma$.
- The growth rate depends on a fractional power of $\eta$ and hence lies between ideal MHD (where the power would be zero) and pure diffusion (where the power would be unity).

The outer solution for the x component of the perturbed magnetic field, B1x, has a discontinuity in its derivative at

$x = 0$ denoted by

$$\Delta' = \frac{B'_{1x}(0)}{B_{1x}(0)}.$$

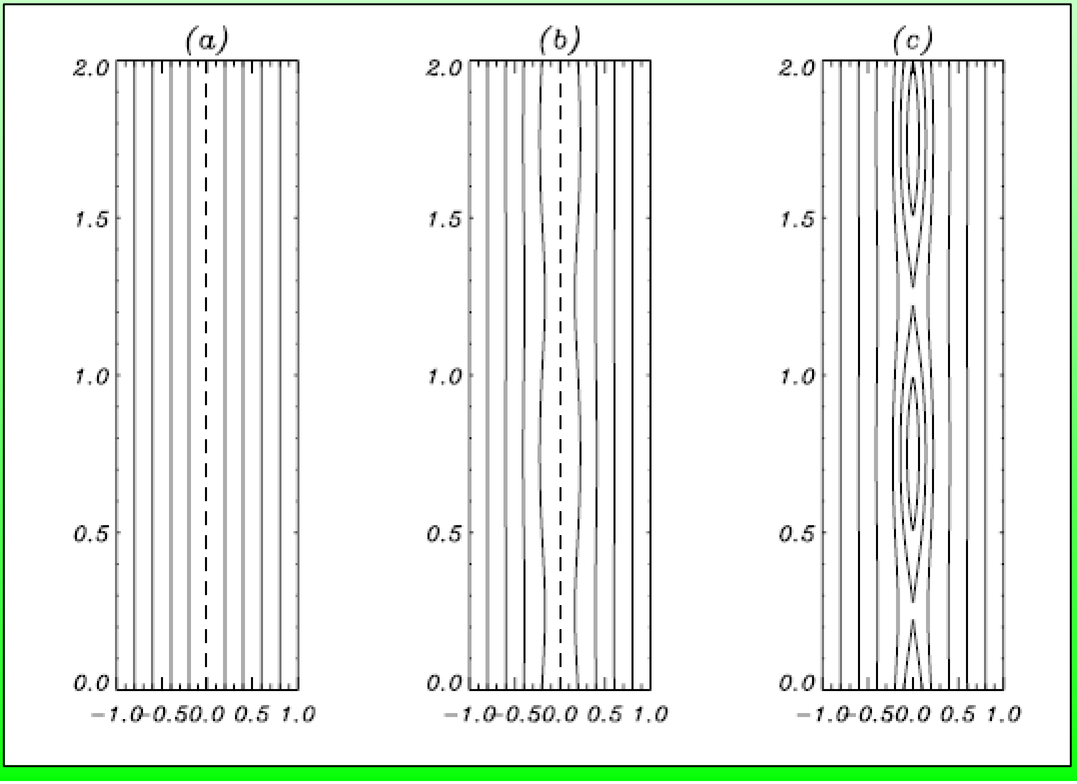
- The growth rate, $\sigma$,

$$\sigma \approx \eta^{3/5} k^{2/5} (\Delta')^{4/5}.$$

- Positive if $\Delta' > 0$ the tearing mode instability condition.
- The width of the boundary layer, $\epsilon$, is

$$\epsilon \approx \eta^{2/5} k^{-2/5} (\Delta')^{1/5}.$$

- In (a) we have the basic equilibrium, where the magnetic field is projected onto the $x - y$ plane.
- Perturb as in (b). If ideal ($\eta = 0$), magnetic pressure would increase where the field is pushed together.
- In a resistive plasma, where field is pushed together has higher current and will diffuse faster than where pulled apart.
- Diffusion of the plasma reduces the current and reduces the build up of magnetic pressure.
- Perturbation also bends field lines and so also magnetic tension force opposes the perturbation.
- Thus, instability if *field diffuses faster than the tension force can straighten*. Tension is smaller for smaller values of $k$.
- $\Delta' > 0$ for smaller values of $k$ (unstable) and $< 0$ (stable) for large values of $k$.



# Thermal Instabilities

$$\frac{\rho^{\gamma}}{\gamma-1}\frac{D}{Dt}\left(\frac{p}{\rho^{\gamma}}\right)=\nabla\left(\kappa\nabla T\right)-\rho^{2}Q(T)+\frac{j^{2}}{\sigma}+H$$

Thermal conduction

Optically thin radiation

Ohmic heating

Coronal heating

## Thermal Instability (optically thin radiation)

- The thermal instability is driven by form of the optically thin radiation.

$$\rho^2 Q(T),$$

where the radiative loss function $Q(T)$ depends strongly on the temperature.

- $Q(T)$ approximated by

$$Q(T) = \chi T^\alpha.$$

- $\chi$ and $\alpha$ depend on temperature. $\alpha$ defines the slope and $\chi$ such that $Q(T)$ continuous.
- Largest values in the transition region.

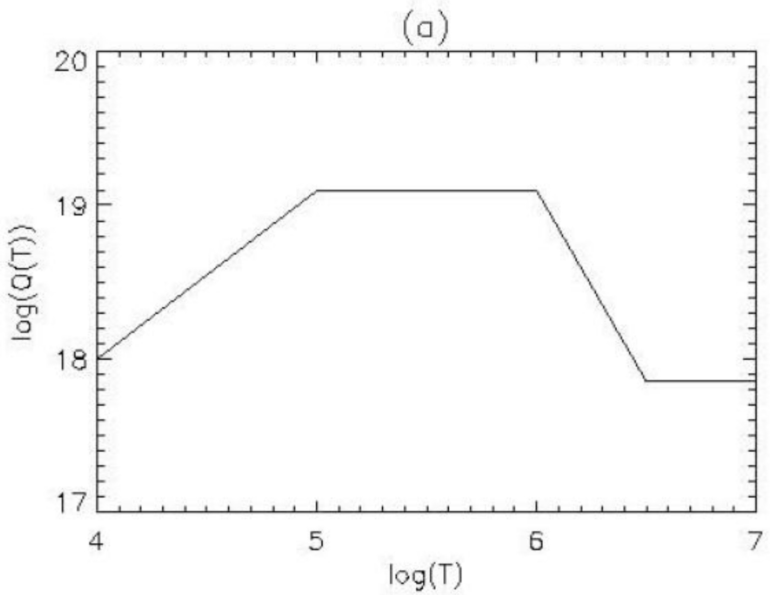



Figure: Optically thin radiative losses as a function of temperature.

- Instability is clear from the shape of this function.
- Consider plasma at $5 \times 10^6$K that cools slightly. The radiation term will now increase and plasma will cool even more. Continues until plasma reaches $10^5$K.
- Define a *radiative timescale*, $\tau_{rad}$, as

$$\frac{1}{\gamma - 1}\frac{Dp}{Dt} \approx \rho^2 \chi T^\alpha, \quad \Rightarrow \quad \frac{p}{\tau_{rad}} = \rho^2 \chi T^\alpha.$$

- At transition region temperatures the radiative timescale is the order of seconds.
- At coronal temperatures it is more like $3,000$ seconds.
- Isobaric thermal instability if $\alpha < 2$.

$$\nabla \cdot (\kappa \nabla T_0) = 0 = \left(\frac{p_0}{\tilde{\mu} R}\right)^2 Q(T_0) - H_0.$$

- $\kappa = \kappa_0 T_0^{5/2}$ thermal conductivity parallel to $\mathbf{B}_0$, $p_0$ constant gas pressure and $H_0$ uniform coronal heating term.
- Linearised equations give equation for $T_1 = T_1 e^{ikx+\sigma t}$ if we assume isobaric perturbations (i.e. $\tau_s \ll \tau_{rad}$).
- The plasma is unstable if

$$\sigma \frac{\gamma p_0}{(\gamma - 1) T_0} = -k^2 \kappa_0 T_0^{5/2} - \rho^2 (\alpha - 2) \chi T_0^{\alpha - 1} > 0.$$

Note that thermal conduction stabilises the instability particularly effective when $k$ large.

# Acknowledgements

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# References

Papers by B. Roberts
Book by E. R. Priest
Book by V. Krishnan
Book by M. Goossens