

PHYS 8120 Plasma Physics and Magnetohydrodynamics

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Topic - Collisions, Deflection and Resistivity

Lecture notes

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1 COLLISION

1.1 UNCHARGED PARTICLES

Consider a long tube of cross section area A , containing incoming gas particles and a test particle inside it as shown in figure 1 below.

Let the test particle of density n move with a velocity V_0 inside the tube. The collision cross-section is given by

$$\sigma = \pi r^2 \quad (1)$$

where r is the radius of the particle.

NOTE: To estimate how long and how far the particle will travel before it collides depends on factors such as the total number of target particles in the tube and size of the target particle. This is a stochastic process which has no simple solution. Hence, such an estimation is currently ignored.

The mean free path (λ) is the shortest distance travelled by a particle in between two collisions. This value of λ depends on the velocity of the particle. Suppose there are N particles, then $N=nA$. So,

$$n\sigma\lambda = 1 \quad (2)$$

this gives the definition for mean free path as,

$$\lambda = \frac{1}{n\sigma} \quad (3)$$

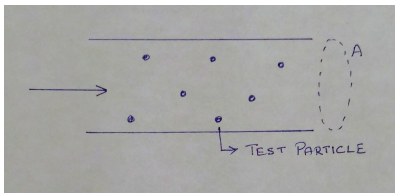


Figure 1: Inflowing gas particles in a tube of cross sectional area A .

The collision frequency is,

$$\nu = \frac{V_0}{\lambda} \quad (4)$$

substituting for V_0 in (4) we get the following expression of collision frequency of an 'uncharged particle',

$$\boxed{\nu = n\sigma V_0} \quad (5)$$

Typical collision time is given by,

$$\tau = \frac{\lambda}{V_0} \quad (6)$$

1.2 CHARGED PARTICLES

To estimate the required collision frequency for charged particles, let the thermal velocity be V_T , because in a plasma the incoming velocity is the thermal velocity. For determining ν and λ , we need an appropriate Coulomb collision cross-section, that is the effective cross section for scattering of a particle through 90° . Looking at the closest distance where a particle energy is comparable to the electrostatic energy we have,

$$\frac{e^2}{r} = \frac{1}{2}m_0V_T^2 \quad (7)$$

So,

$$r = \frac{2e^2}{m_0V_T^2} \quad (8)$$

In principle, the difference between a charged and an uncharged particle interaction is that the former has an infinite range and is limited to the Debye length (λ_D) i.e. even if r is very large, there will be a slight interaction with a small deflection only. For most plasma, the value of r is much smaller than the Debye length.

The most simple approximation for plasma interaction is

$$\sigma = \pi\lambda_D^2 \quad (9)$$

2 DEFLECTION

Referring to figure 2, considering that most interactions between electrons and protons are very small deflections in the velocity and that the centre of gravity is in proton, the next step is to calculate deflection angle (θ) as a function of impact parameter (b) and impact velocity V_0 based on Kepler's gravitational problem. We know the momentum equation for an electron,

$$m_e\ddot{\vec{r}} = -\frac{e^2}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r} \quad (10)$$

We also know that the typical incoming velocity of an electron is the thermal velocity, but there is an extra velocity due to currents because the electrons move with respect to protons. However, this additional velocity can be ignored as it is very small.

NOTE: The following expression can be used to calculate the thermal velocity from the temperature of the electron.

$$\frac{1}{2}m_eV_{th}^2 = \frac{3}{2}kT_e \quad (11)$$

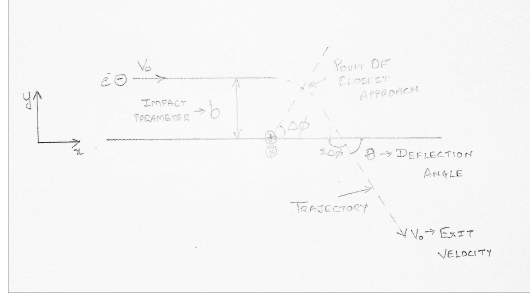


Figure 2: Deflection in electron path approaching a proton at a distance of closest approach.

With the knowledge of two bodies moving under gravitational attraction, we know

1. All the motions are in the plane (x,y) and z=0
2. Angular momentum is conserved i.e. $L_Z = bV_0 = r^2\dot{\phi}$
3. Energy is conserved i.e.

$$\frac{m_e}{2}(r\dot{r}^2 + r^2\dot{\phi}^2) - \frac{e^2}{4\pi\epsilon_0 r} = \frac{m_e}{2}V_0^2 \quad (12)$$

The equations of motion in cylindrical coordinates are

$$m_e(\ddot{r} - r\dot{\phi}^2) = -\frac{e^2}{4\pi\epsilon_0 r^2} \quad (13)$$

$$m_e \frac{d(r^2\dot{\phi})}{dt} = 0 \quad (14)$$

We now have a system of two variables that are first order time derivatives, but nonlinear in nature. The solutions for \dot{r} and $\dot{\phi}$ are,

$$\dot{r} = \pm \left\{ V_0^2 + \frac{2e^2}{4\pi\epsilon_0 r m_e} - \frac{b^2 V_0^2}{r^2} \right\}^{\frac{1}{2}} \quad (15)$$

and

$$\dot{\phi} = \frac{bV_0}{r^2} \quad (16)$$

In (15), when the electron is coming in toward the proton, as the distances are decreasing, we get a minus sign for the first part of the trajectory. After it passes the proton, it will kick off and r will increase again in the second half of the trajectory (refer figure 2). Due to symmetry all quantities are conserved.

We can eliminate dt from the (15) and (16) as time does not appear explicitly on the RHS. So,

$$d\phi = -\frac{\frac{dr}{r^2}}{\left\{ \frac{1}{b^2} + \frac{e^2}{2\pi\epsilon_0 m_e b^2 V_0 r} - \frac{1}{r^2} \right\}^{\frac{1}{2}}} \quad (17)$$

Let $y = \frac{b}{r}$ such that $dy = -\frac{b}{r^2} dr$. Now, defining a dimensionless quantity Q such that,

$$Q \equiv \frac{e^2}{4\pi\epsilon_0 m_e V_0^2 b} \quad (18)$$

we get,

$$d\phi = \frac{dy}{\{1 - 2Qy - y^2\}^{\frac{1}{2}}} \quad (19)$$

As the electron travels, it gets attracted and changes its course and moves out to infinity. In doing so, there is an angle of deflection arising which we are interested to know. To solve for the change in deflection angle we assume the following boundary conditions;

1. $y = \frac{b}{r}$
2. $t = 0$
3. $r = \infty (\implies y_0 = 0)$
4. For the closest approach $r = r_{min}$, we get $y = y_{max}$ such that $t = t_{max}$

So,

$$\phi = \int_{y_{min}=0}^{y_{max}} \frac{dy}{\{1 - 2Qy - y^2\}^{\frac{1}{2}}} \quad (20)$$

Notice that, $y = y_{max}$ when $\frac{dy}{d\phi} = 0$

$$\implies (1 - 2Qy_{max} - y_{max}^2) = 0$$

Substituting,

$$\sin \psi = \frac{Q + y}{(1 + Q^2)^{\frac{1}{2}}} \quad (21)$$

So that,

$$d \sin \psi = \cos \psi = \frac{dy}{(1 + Q^2)^{\frac{1}{2}}} \quad (22)$$

Therefore on solving we get,

$$\boxed{(\phi_{max} - \phi_0) = \int_{\psi_0}^{\psi_{max}} d\phi = (\psi_{max} - \psi_0)} \quad (23)$$

Now for the boundary conditions,

1. At $r = 0$, we get $y = 0 \implies \sin \psi_0 = \frac{Q}{(1+Q^2)^{\frac{1}{2}}}$
2. $y = y_{max} \implies \sin \psi_{max} = \frac{(Q+y_{max})}{(1+Q^2)^{\frac{1}{2}}}$

$$\text{We have, } (1 - 2Qy_{max} - y_{max}^2) = 0 \implies 1 + Q^2 - (Q + y_{max})^2 = 0$$

Therefore, $\sin \psi_{max} = 1$ gives $\psi_{max} = \frac{\pi}{2}$

Also, from condition (1) above we get, $\psi_0 = \arctan Q$

On solving we get the following expression for deflection angle

$$\boxed{\Delta\phi = \frac{\pi}{2} - \arctan Q} \quad (24)$$

where $\Delta\phi$ is the change in position angle of the electron between $r_0 = \infty$ and $r = r_{min}$ (closest approach). When there is no deflection at all i.e. if the electron particle goes straight through then, by definition, $\Delta\phi = \pi$.

But we are interested to know the angle θ as shown in the figure 2. Then

$$\boxed{\theta = \pi - 2\Delta\phi = 2 \arctan Q} \quad (25)$$

Typically, in mks units, $Q = \frac{-e^2}{m_e v_0^2 b}$

NOTE: We know that the impact parameter b cannot be larger than the Debye length. So, for the smallest value of b we have large value of Q . That indicates θ is small!

By estimating the deflection θ , we can know the amount of momentum the electron is losing. For small deflections, the momentum lost is also small. The net thermal velocity V_0 is conserved.

3 RESISTIVITY

We have an electron moving in an electric field and losing momentum by deflection of small angle.

$$m_e \frac{d\vec{V}_{e\parallel}}{dt} = -e\vec{E} - m_e \nu_{i,e} \Delta V_{e\parallel} \quad (26)$$

where $\nu_{i,e}$ is the electron-proton collision frequency, and $V_{e\parallel}$ is the thermal velocity of an electron relative to the protons parallel to the electric field.

Since the current density is given by

$$\vec{j} = -en_e V_{e\parallel} \quad (27)$$

clearly the collision frequency in (26) is proportional to the current density so that we can combine (26) and (27) to get,

$$\vec{E} = \frac{m_e \nu_{i,e}}{e^2 n_e} \frac{\Delta V_{e\parallel}}{V_{e\parallel}} \vec{j} \quad (28)$$

In the zeroth order approximation, we assume $\Delta V_{e\parallel} = V_{e\parallel}$ so that,

$$\vec{E} = \eta \vec{j} \quad (29)$$

where

$$\eta = \frac{m_e \nu_{i,e}}{e^2 n_e} \quad (30)$$

Clearly, from (29), η is the electrical resistivity of the plasma.