Lec 12 Note : 03/03/2025

Energy Equations

Starting with the induction equation.

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{V} \times \vec{B}) + \vec{\nabla} \times n\vec{j}$$

Taking dot product with $\vec{B}$;

$$\frac{\partial \vec{B}}{\partial t} \cdot \vec{B} = \vec{\nabla} \times (\vec{V} \times \vec{B}) \cdot \vec{B} + (\vec{\nabla} \times n\vec{j}) \cdot \vec{B}$$

using the Ohm's law for the right side of the equation;

$$\Rightarrow \quad \frac{1}{2}\frac{\partial B^2}{\partial t} = (\vec{\nabla} \times \vec{E}) \cdot \vec{B}$$

Using vector identities;

$$\frac{1}{2}\frac{\partial B^2}{\partial t} = \vec{\nabla} \cdot (\vec{E} \times \vec{B}) + \vec{E}\,(\vec{\nabla} \times \vec{B})$$

$$\boxed{\frac{1}{2}\frac{\partial B^2}{\partial t} = \vec{\nabla} \cdot \vec{P} + \vec{E} \cdot \vec{J}}$$

pointing flux

Now using the momentum equation:

$$\rho \frac{d\vec{v}}{dt} = \vec{j} \times \vec{B} + \vec{\nabla} p - \rho g$$

and

$$\rho \frac{d\vec{v}}{dt} = \rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{\nabla}(\vec{\nabla} \cdot \vec{v})$$

$$\Rightarrow \rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) = \vec{j} \times \vec{B} + \vec{\nabla} p - \rho g$$

Taking dot product with $\vec{v}$:

$$\Rightarrow \rho \frac{\partial \vec{v}}{\partial t} \cdot \vec{v} + \rho \vec{v} \cdot \rho \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) = \vec{v} \cdot (\vec{j} \times \vec{B}) + \vec{v} \cdot (\vec{\nabla} p) - \vec{v} \cdot \rho g$$

Finally: Energy equation for plasma

$$\boxed{\frac{1}{2} \rho \frac{\partial v^2}{\partial t} + \left(\frac{\vec{\nabla} \rho v^2}{2}\right) \cdot \vec{v} = \underbrace{\vec{v} \cdot (\vec{j} \times \vec{B})}_{\text{work of lorentz force}} + \underbrace{\vec{v} \cdot (\vec{\nabla} p)}_{\text{work done by pressure}} - \underbrace{\vec{v} \cdot \rho g}_{\text{Work done by gravity}}}$$

So we have;

3 equations from momentum equation.

3 equations from induction equation.

1 continuity equation.

1 energy equation.

Using,

$p = 2\rho RT$ and $\vec{\nabla} \times \vec{B} = \vec{j}$ we can replace

T and $\vec{j}$ in above equations,

which gives us 8 equations to solve

for the 8 variables:

$\vec{V} \rightarrow 3$ components

$\vec{B} \rightarrow 3$ components

$\rho \rightarrow 1$ component

$p \rightarrow 1$ component