Plasma-physics - 2025-1-29

→ MHD equation and ohm's law.

Magneto-hydrodynamics (MHD) couples maxwell's equations of electromagnetism with hydro-dynamics to describe the macroscopic behaviours of conducting fluids such as plasmas.

We have from maxwell's equations.

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial B}{\partial t} \quad \text{—(i)}$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

or. $\vec{\nabla}\cdot\vec{E} = 4\pi q = 0$, for $L > \lambda_D$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c}\vec{J} \quad \text{—(1)}$$

↗ This term is much smaller than other two terms

$$\frac{1}{c}\frac{\partial E}{\partial t} \simeq \frac{E}{c}\frac{v_0}{l_0} = \frac{1}{c}\frac{1}{t_0} \rightarrow \text{Very small.}$$

where, l_0 = length scale

V_0 = Velocity.

Using ohm's law

$E + \frac{\vec{V} \times \vec{B}}{c} = 0$ [ideal plasma, no resistivity]

$E + \frac{\vec{V} \times \vec{B}}{c} = \eta j$, (non-ideal)
$\eta \to$ resistivity

[Collision between electrons and protons in opposite way slow down motion producing η]

So, $\frac{1}{c} \frac{\partial E}{\partial t} = \frac{1}{c} \frac{1}{t_0} = \frac{|V_0|}{c} \times \frac{B|V_0|}{c} = \frac{V_0^2 \vec{B}}{c^2}$ — (*)

$\frac{1}{c} \frac{\partial E}{\partial t} = \frac{V_0^2 \vec{B}}{c^2}$ — (*)

i.e. comparision to B, $\vec{E}$ is very small.

In Vacuum, there is no current, $\frac{4\pi}{c} j = 0$.

The eqn (1) can be written as,

$\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$

$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{\nabla} \times \vec{E}}{\partial t}$

$= -\frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$ [Using eqn -i]

$$\therefore \nabla^2 B = \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2}$$

$\therefore \boxed{\Delta B = \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2}}$ — (2)

Again from eqn (1) neglecting small term $\frac{1}{c}\frac{\partial E}{\partial t}$ and taking only significant term

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} \quad \text{— (3)}$$

Considering ideal plasma,

$$\vec{E} + \frac{\vec{V}\times\vec{B}}{c} = 0$$

So, $\vec{E} = -\frac{\vec{V}\times\vec{B}}{c}$ — (4)

For non ideal plasma, $\vec{E} + \frac{\vec{V}\times\vec{B}}{c} = \eta\vec{J} = c\,\frac{\vec{\nabla}\times\vec{B}}{4\pi}$

Now, For ideal case

$$\vec{\nabla}\times\vec{E} = -\nabla\times\left(\frac{\vec{V}\times\vec{B}}{c}\right) = -\frac{1}{c}\frac{\partial B}{\partial t}$$

non-ideal,

$$\frac{\partial B}{\partial t} = \nabla\times(\vec{V}\times\vec{B}) + \frac{\eta}{\mu_0}\nabla^2 B.$$

$\downarrow$ convection, diffusion

$$\nabla\times\vec{E} = -\nabla\times\left(\frac{\vec{V}\times\vec{B}}{c}\right) - \nabla\times\eta\vec{J} = \frac{1}{c}\frac{\partial B}{\partial t}$$

$$+\frac{1}{c}\frac{\partial B}{\partial t} + \vec{\nabla}\times\vec{B} = 0 \quad \text{— (*)}$$

Momentum equation for plasma.

momentum equation of electron for plasma

$$m_e n_e \frac{d\vec{v_e}}{dt} = nq\left(\frac{\vec{V_e} \times \vec{B}}{c}\right) + nq\vec{E} - \vec{\nabla} P_e \quad \text{— (i)}$$

$$m_p n \frac{d\vec{v_p}}{dt} = nq\left(\frac{\vec{V_p} \times \vec{B}}{c}\right) - nq\vec{E} - \vec{\nabla} P_p \quad \text{— (ii)}$$

where, $(P_e + P_e) = P$ (gas pressure)

$$q_e n_e V_e + (-q_p n_p v_p) = \vec{J} \quad \text{— (iii)}$$

adding eqn (i) and (ii)

$$n\left(m_e \frac{d\vec{v_e}}{dt} + m_p \frac{d\vec{v_p}}{dt}\right) = nq\left(\frac{\vec{v_e}}{c} \times \vec{B} - \frac{\vec{V_p}}{c} \times \vec{B}\right) - \nabla(P_e + P_p)$$

$$\text{or, } n\left(m_e \frac{d\vec{v_e}}{dt} + m_p \frac{d\vec{v_p}}{dt}\right) = nq\,(v_e - v_p) \times \frac{\vec{B}}{c} - \nabla P$$

$$\therefore n\left(m_e \frac{d\vec{v_e}}{dt} + m_p \frac{d v_p}{dt}\right) = \frac{\vec{J} \times \vec{B}}{c} - \vec{\nabla} \vec{P} \quad \text{— (iv)}$$

using eqn (iii),

$\frac{\vec{J} \times \vec{B}}{c} \rightarrow$ Lorentz force in plasma.

Now, Let us define average velocity of plasma as,

$$n\vec{v} = \left(\frac{n_e m_e \times \vec{v}_e + n_p m_p \times \vec{v}_p}{m_e + m_p} \right)$$

Taking m_p common on both denum

$$n\vec{v} = \left(\frac{n_e \frac{m_e}{m_p} \vec{v}_e + n\vec{v}_p}{\frac{m_e}{m_p} + 1} \right)$$

Since, $m_e << m_p$, $\frac{m_e}{m_p} << 1$

Therefore, $n\vec{u} = n_p \vec{v}_p$ — (V)

Thus, velocity of proton can be considered as velocity of plasma.

From eqn (iv) changes to the form,

$$n m_p \frac{dv_p}{dt} = n_p V$$

So, $\rho \frac{d\vec{v}}{dt} = \frac{\vec{J} \times \vec{B}}{c} - \vec{\nabla} p$ — (vi)

This is the momentum equation for plasma.

To Derive ohm's law

$$m_e n \frac{d\vec{v}_e}{dt} = nq\left(\frac{\vec{V}_e \times \vec{B}}{c}\right) + nqE - \vec{\nabla} P_e \quad \text{— (i)}$$

$$m_p n \frac{dv_p}{dt} = nq\,(\vec{V}_p \times \vec{B}) - nq\vec{E} - \vec{\nabla}\vec{P}_p \quad \text{— (ii)}$$

Substracting eqn (i) and (ii)

$$n(m_e \frac{dv_e}{dt} - m_p \frac{dv_p}{dt}) = nq\left(\frac{\vec{V}_e \times \vec{B}}{c} - \frac{\vec{V}_p \times \vec{B}}{c}\right) + 2q\vec{E} - \vec{\nabla}P_e + \vec{\nabla}P_p$$

$$[\because P_e = P_p]$$

$$\Rightarrow n\left(m_e \frac{d\vec{v}_e}{dt} - m_p \frac{dv_p}{dt}\right) = nq\,(v_e + v_p) \times \frac{1}{c} \times \vec{B}) + 2q\vec{E}.$$

$m_e << m_p$ so, first term is negleted.

$$\Rightarrow -n m_p \frac{dv_p}{dt} = nq\,(v_e + v_p)\frac{\times \vec{B}}{c} + 2q\vec{E}$$

$$\Rightarrow -n m_p \frac{dv_p}{dt} = nq\,(J/nq + v_p + v_p) \times \vec{B}/c + 2q\vec{E}$$

$$\int \frac{dv_p}{dt} = \vec{J}\frac{\times \vec{B}}{c} + 2nq\frac{\vec{v}_p \times \vec{B}}{c} + 2q\vec{E} \quad \text{— (iii)}$$

using plasma, momentum eqn — ⊗

$$\rho \frac{du}{dt} = \frac{\vec{J} \times \vec{B}}{c} - \vec{\nabla} p \quad \text{— (iv)}$$

Now, eqn (iii) changes to the form

$$\frac{\vec{J} \times \vec{B}}{c} - \nabla p = \frac{\vec{J} \times \vec{B}}{c} + 2nq \frac{\vec{v}_p \times \vec{B}}{c} + 2q\vec{E}$$

$$\frac{n\vec{v} \times \vec{B}}{c} + \vec{E} = 0 \quad \text{— (v)}$$

This is ohm's law in plasma. [MHD] — (*)