02/10/2025 (Lec 07) — Enosh

Frozen-in plasma Theory of MHD

For ideal case,

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla}(\vec{\nabla}\times\vec{B}) \; ; \quad (\eta = 0)$$

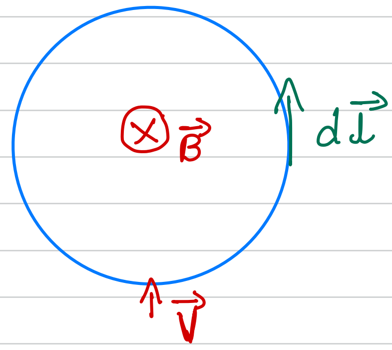
Taking integral of both sides,

$$\iint \frac{\partial \vec{B}}{\partial t}\, d\vec{\sigma} = -\iint \vec{\nabla}(\vec{\nabla}\times\vec{B})\, d\vec{\sigma}$$

Using Stoke's theorem;

$$\Rightarrow \iint \frac{\partial \vec{B}}{\partial t}\, d\vec{\sigma} = -\oint \vec{\nabla}\times\vec{B}\; d\vec{l}$$

in a tube,



$$\Rightarrow \iint \frac{\partial \vec{B}}{\partial t} \, d\vec{O} = - \int V_{\perp} B \, dl \quad - (1)$$

We know that,

$$\phi \equiv \iint \vec{B} \cdot d\vec{O} \qquad (\phi \rightarrow \text{flux})$$

Also,

$$\frac{d\phi}{dt} = \underbrace{\frac{\partial \phi}{\partial t}}_{\iint \frac{\partial \vec{B}}{\partial t} d\vec{O}} + \underbrace{\int V_{\perp} B_{l} \, dl}_{\text{from } (1):\ -\iint \frac{\partial \vec{B}}{\partial t} d\vec{O}}$$

Finally we get,

$$\boxed{\frac{d\phi}{dt} = 0}$$

* No flux change.
* Because the plasma is moving along with magnetic field. (Frozen-in)

Also we can prove the plasma is moving with the flux in the following way:

Assume,

$$\vec{B} = B_z \hat{z}$$

For an electron,

$$m_e \frac{d\vec{V_e}}{dt} = -\vec{e}\left(\vec{V_e} \times \vec{B}\right)$$

1) $$\frac{d\vec{V_z}}{dt} = 0$$

2) $$m_e \frac{d\vec{V_x}}{dt} = -e\left(V_y B_z\right)$$

3) $$m_e \frac{d\vec{V_y}}{dt} = -e\left(V_x B_z\right)$$

$$m_e \frac{d\vec{V_e}}{dt} \cdot \vec{V_e} = \underbrace{-e\left(\vec{V_e} \times \vec{B}\right) \cdot \vec{V_e}}_{0}$$

Vorticity ($\vec{\omega}$) preservation

Navier – Stokes equation,

$$\rho \frac{d\vec{V}}{dt} = \rho \frac{\partial \vec{V}}{\partial t} + \rho(\vec{V} \cdot \vec{\nabla})\vec{V} = -\vec{\nabla} p - \rho \vec{\nabla} \psi$$

Since fluid is incompressible,

$$\rho = \text{constant.}$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} = 0$$

So from continuity equation,

$$\underbrace{\frac{\partial \rho}{\partial t}}_{0} + \vec{\nabla} \cdot (\rho u) = 0$$

$$\Rightarrow \quad \vec{\nabla} \cdot \rho \vec{V} = 0$$

$$\Rightarrow \quad \rho (\vec{\nabla} \cdot \vec{V}) = 0$$

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{V} = 0$$

So we get,

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho (\vec{V} \cdot \vec{\nabla}) \vec{V} = 0$$

Taking curl;

$$\rho \left(\vec{\nabla} \times \frac{\partial \vec{V}}{\partial t} \right) + \rho \left(\vec{\nabla} \times (\vec{V} \cdot \vec{\nabla}) \vec{V} \right) = 0$$

vorticity is defined as,

$$\boxed{\vec{\omega} \equiv \vec{\nabla} \times \vec{V}}$$

$$\Rightarrow \quad \frac{\partial \vec{\omega}}{\partial t} + \vec{\nabla} \times (\vec{V} \cdot \vec{\nabla}) \vec{V} = 0 \quad \text{--- } (1)$$

using,

$$(\vec{V} \cdot \vec{\nabla}) \vec{V} = \frac{1}{2} \vec{\nabla} (\vec{V} \cdot \vec{V}) - \vec{V} \times \underbrace{(\vec{\nabla} \times \vec{V})}_{\vec{\omega}}$$

Taking Curl,

$$\vec{\nabla} \times (\vec{V} \cdot \vec{\nabla}) \vec{V} = \underbrace{\frac{1}{2} \vec{\nabla} \times \left(\vec{\nabla} (\vec{V} \cdot \vec{V}) \right)}_{0} - \vec{\nabla} \times (\vec{V} \times \vec{\omega})$$

$$\Rightarrow \quad \vec{\nabla} \times (\vec{V} \cdot \vec{\nabla}) \vec{V} = - \vec{\nabla} \times (\vec{V} \times \vec{\omega})$$

Finally from 1 ;

$$\boxed{\frac{\partial \vec{\omega}}{\partial t} - \vec{\nabla} \times (\vec{V} \times \vec{\omega}) = 0}$$