2025/02/17 - Day 08

Amphere's law : MHD : $\nabla \times B = \frac{4\pi}{c} \bar{J}$ 8

Stock theorem :

for a wire

$$\iint \nabla \times B \, dA = \frac{4\pi}{c} \underbrace{\iint \bar{J} \, d\bar{A}}_{I}$$

Stock theorem $\longrightarrow$ $B_\phi 2\pi r = \frac{4\pi}{c} I$

$$B_\phi = \frac{2I}{cr}$$

Plasma $\beta = \frac{E_{thermal}}{E_{magnetic}}$

$= \frac{8\pi P}{B^2}$

$E_{thermal} = P$; $E_{magnetic} = \frac{B^2}{8\pi}$ (direve later)

for takamak $\beta < 1$ $\because$ $E_{th} < E_{mag}$

takamak buit for containt plasma in magnetic feild. $\therefore$ E_{mag} is stronge.

force equation : $F = \bar{J} \times \bar{B} + \nabla p$

$$\frac{B^2}{\ell} > \frac{P}{\ell}$$

if $F = 0 \longrightarrow \bar{J} \times \bar{B} = 0$ $\because$ can't find ~~numerical~~ analatical solutions since, $(\nabla \times \bar{B}) \times \bar{B} = 0$

$\therefore$ need to use $\underbrace{\text{numerical}}$ Simulation

$\bar{J} = \alpha \cdot \vec{r}\, \bar{B}$ $\because$ J is parallel

α is a scalar

$\nabla J = \underbrace{\nabla \cdot (\nabla \times \bar{B})} = 0.$

$\therefore$ $\nabla \alpha (\bar{r} \cdot \bar{B}) = 0$

$\nabla \alpha \cdot \bar{B} + \alpha \cdot \nabla \bar{B} = 0$

$\nabla \times \bar{B} = 0$

when the magnetic feild is Strong $F \approx 0$

define M-feild,

$B = \bar{\nabla} \gamma \times \nabla \delta$; γ and δ are Scalar functions

$\nabla B = \underbrace{\bar{\nabla} \cdot (\bar{\nabla} \gamma \times \bar{\nabla} \delta)}_{0}$ note book

$\therefore$ Simplify the Simulation

but $\bar{\nabla} \times \bar{B} = \bar{\nabla} \times (\bar{\nabla} \gamma \times \bar{\nabla} \delta) \longrightarrow$ will leads to complex ~~eq~~ form

$\therefore$ not a good way to solve.

need to find Solutions based on Symmetry of B.C.

from book : (M-feild in Cylindrical coordinates)

$$(\nabla \times B)_r = \frac{1}{r}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z} \quad ; \quad (\nabla \times B)_z = \frac{1}{r}\frac{\partial}{\partial r}(r B_\phi) - \frac{1}{r}\frac{\partial B_r}{\partial \phi}$$

$$(\nabla \times B)_\phi = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}$$

$$(\nabla \cdot B) = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \cancel{\frac{1}{r}\frac{\partial B_\phi}{\partial \phi}}^{0} \cdot \cancel{\frac{\partial B_z}{\partial z}}^{0} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r)$$

$\because$ no change in ϕ, z direction
$\because$ Cylindrical Symmetry

we know,

& $\nabla \cdot B = 0$

$$\therefore 0 = \frac{1}{r}\frac{\partial}{\partial r}(rB_r)$$

$$0 = \frac{\partial}{\partial r}(rB_r)$$

$K = rB_r$; K is a constant.

$\frac{K}{r} = B_r$ but at $r = 0$ ~~$B_r = 0$~~ (It can't be correct.)

$\therefore K = 0$

$\therefore B_r = 0$ ← let's use this with $\nabla \times B$

$$\therefore \underbrace{(\nabla \times B)_\phi}_{J_\phi} = -\frac{\partial B_z}{\partial r} = J_\phi$$

$$(\nabla \times B)_r = \frac{1}{r}\cancel{\frac{\partial B_z}{\partial \phi}}^{0} - \cancel{\frac{\partial B_\phi}{\partial z}}^{0}$$ due to cylindrical Symmetry

$$\therefore (\nabla \times B)_r = 0 = J_r$$

$$(\nabla \times B)_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\phi) = J_z$$

to find force

$$F = J \times B = J_\phi B_z - J_z B_\phi = F_r$$; $F_z = 0$ and $F_\phi = 0$ are zero since B_r and $J_r = 0$

assumption $F = 0$ (force free feild)

$$\therefore F_r = 0 = -\frac{\partial B_z}{\partial r}\cdot B_z - \frac{1}{r}\frac{\partial}{\partial r}(rB_\phi)\cdot B_\phi$$

$$0 = +\frac{\partial}{\partial r}\left(\frac{B_z^2}{2}\right) + \frac{1}{r}B_\phi^2 + \frac{r}{r}\frac{\partial B_\phi}{\partial r}\cdot B_\phi$$

$$0 = +\frac{\partial}{\partial r}\left(\frac{B_z^2}{2}\right) + \frac{B_\phi^2}{r} + \frac{r}{r}\frac{\partial}{\partial r}\left(\frac{B_\phi^2}{2}\right)$$

$$0 = \frac{\partial}{\partial r}\underbrace{\left(\frac{B_z^2}{2} + \frac{B_\phi^2}{2}\right)}_{\frac{B^2}{2}} + \frac{B_\phi^2}{r}$$

$\because B = \cancel{B_r}^{0}\hat{r} + B_\phi\hat{\phi} + B_z\hat{z}$

$B^2 = B_\phi^2 + B_z^2$

$$0 = \frac{\partial}{\partial r}B^2 + \frac{2B_\phi^2}{r}$$

there can be many solution for this

ex: $B_r = 0$

$$B_z = \frac{B_0}{1 + b^2r^2}$$

$$B_\phi = \frac{B_0 br}{1 + b^2r^2}$$

Proof :

$$\text{R.H.S} = \frac{\partial}{\partial r}B^2 + \frac{2B_\phi^2}{r}$$

$$= \frac{\partial}{\partial r}\left(\frac{B_0^2 b^2 r^2}{(1+b^2r^2)^2} + \frac{B_0^2}{(1+b^2r^2)^2}\right) + \frac{2B_0^2 b^2 r^2}{r(1+b^2r^2)^2}$$

(golden - hoyle Solutions) / Lundquist feild Solutions

$$R.H.S \equiv B_0^2 \frac{\partial}{\partial r}\left(\frac{\cancel{(b^2r^2+1)}}{(1+b^2r^2)^2}\right) + \frac{2.\, B_0^2\, b^2 \cancel{r}}{\cancel{r}(1 + b^2r^2)}$$

$$\equiv B_0^2\left(- \frac{1 \times b^2 . \cancel{2} r}{(1+\cancel{b^2r^2})^2}\right) + \frac{2 B_0^2 \cancel{b^2} r}{(1 + b^2r^2)^2}$$

$$\equiv 0 \equiv L.H.S$$

H.W. (tips): $f(r) = B^2$ (need to be positive) - define B^2 as a function

$$B_\phi^2 = \frac{-r}{2}\frac{\partial f}{\partial r} \quad \text{and} \quad B_z^2 = B^2 - B_\phi^2$$

for example (previous case)

$$f(r) = B^2 = B_\phi^2 + B_r^2 + B_z^2 = \frac{B_0^2}{(1+b^2r^2)}$$

* Solar flares are due to lorentz force.
When there is no equlibrium solution