MHD Waves and Stability

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Let's choose the coordinate system so that, B_0 is aligned with the z-axis.

$$\vec{B} = \overrightarrow{B_0} = B_0\hat{z}$$

Consider equilibrium, static, homogenous, satisfies MHD equations,

- ρ_0 – constant.
- No plasma motion of zeroth order $\left|\overrightarrow{V_0}\right| = 0$
- Temperature is constant.
- Ideal metallic behavior, $\eta = 0$
- Ignore the gravity, $|\vec{g}| = 0$

Introduce small disturbance to the magnetic field, so that,

perturbations ≪ equilibrium

$$\left|\overrightarrow{B_1}\right| \ll \left|\overrightarrow{B_0}\right|$$

$$p = p_0 + p_1$$

For adiabatic perturbation,

$p_1 = C_s^2\rho_1$ with sound speed $C_s^2 = \gamma p_0/\rho_0 \;\rightarrow$ Constant

Linearize all equations keeping only linear terms using Taylor expansion and maintain only the first order terms. (product of two perturbation terms is small and can be neglected.)

$$\rho_0\frac{\partial\overrightarrow{V_1}}{\partial t} + \rho_1\frac{\partial\overrightarrow{V_0}}{\partial t} + \rho\overrightarrow{V_1}.\left(\vec{\nabla}\overrightarrow{V_1}\right) = -\vec{\nabla}p + \left(\overrightarrow{J_0}\times\overrightarrow{B_1}\right) + \left(\overrightarrow{J_1}\times\overrightarrow{B_0}\right) \;\longrightarrow (01)$$

- Since $|\overrightarrow{V_0}| = 0, \rightarrow \rho_1 \frac{\partial \overrightarrow{V_0}}{\partial t} = 0$
- We neglect the second order term, $\rho \overrightarrow{V_1}.(\overrightarrow{\nabla} \overrightarrow{V_1})$
- Since $\overrightarrow{J_0} = 0, \rightarrow \overrightarrow{J_0} \times \overrightarrow{B_1} = 0$
- $p = p_0 + p_1$ and p_0 is a constant

Then the equation (01) can be written as,

$$\rho_0 \frac{\partial \overrightarrow{V_1}}{\partial t} = -\overrightarrow{\nabla} p_1 + (\overrightarrow{J_1} \times \overrightarrow{B_0}) \quad \longrightarrow (02)$$

Also, $\overrightarrow{J_1} = \overrightarrow{\nabla} \times \overrightarrow{B_1}$ and $p_1 = C_s^2 \rho_1$, by substituting these relations to equation (02), we get,

$$\rho_0 \frac{\partial \overrightarrow{V_1}}{\partial t} = -C_s^2 \overrightarrow{\nabla} \rho_1 + \left((\overrightarrow{\nabla} \times \overrightarrow{B_1}) \times \overrightarrow{B_0}\right) \quad \longrightarrow (03)$$

From the induction equation,

$$\frac{\partial \overrightarrow{B_1}}{\partial t} = \overrightarrow{\nabla} \times (\overrightarrow{V_1} \times \overrightarrow{B_0}) \quad \longrightarrow (04)$$

Now we have two linear partial differential equations, (03) and (04).

Solutions,

$$\overrightarrow{B_1}(t, \vec{r}) = \overrightarrow{B_c}\, e^{i(\omega t - \vec{k}.\vec{r})}$$

$$\overrightarrow{V_1}(t, \vec{r}) = \overrightarrow{V_c}\, e^{i(\omega t - \vec{k}.\vec{r})}$$

$$\rho_1(t, \vec{r}) = \rho_c\, e^{i(\omega t - \vec{k}.\vec{r})}$$

Where B_c, V_c, and ρ_c are constants. Taking the curl and grad of above the solutions, we can have following relations,

$$\overrightarrow{\nabla} \times \overrightarrow{B_1} = i\vec{k} \times \overrightarrow{B_1}$$

$$\overrightarrow{\nabla}.\overrightarrow{B_1} = i\vec{k}.\overrightarrow{B_1}$$

$$\overrightarrow{\nabla} \rho_1 = i\vec{k} \rho_1$$

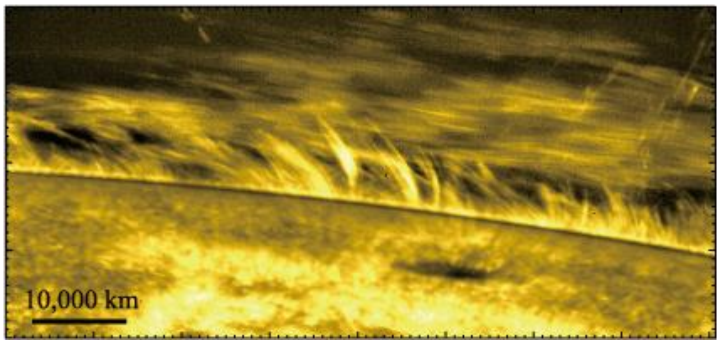
By plugin $\overrightarrow{B_1}(t, \vec{r})$ to the equation (04),

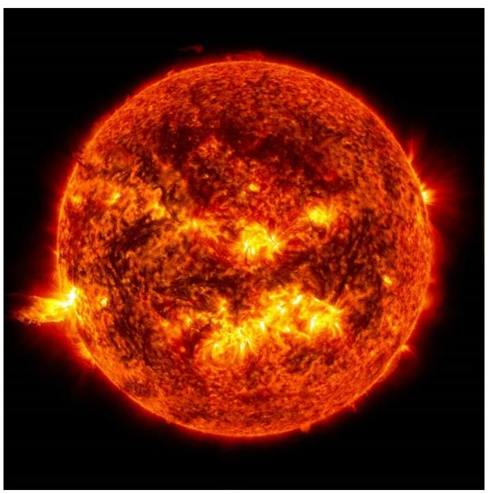
$$i\omega \overrightarrow{B_c} = \overrightarrow{\nabla} \times (\overrightarrow{V_1} \times \overrightarrow{B_0})$$

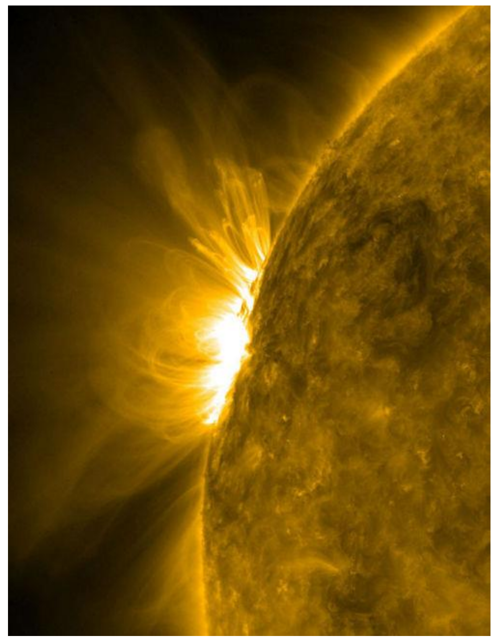
Alfvén wave

In plasma physics, an Alfvén wave, named after Hannes Alfvén, is a type of magneto-hydrodynamic wave in which ions oscillate in response to a restoring force provided by an effective tension on the magnetic field lines. [01]

- $\rho_1 = 0$
- $\vec{k} \parallel \vec{B_0}$
- $\vec{V_1} \perp \vec{B_0}$
- $\vec{B_1} \perp \vec{B_0}$







Alfvén waves in Solar corona

Alfvén Velocity V_A,

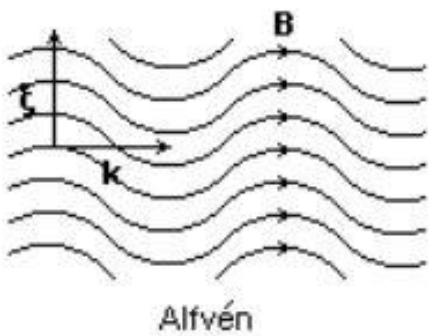
$$V_A = \frac{B_0}{\sqrt{4\pi\rho_0}}$$

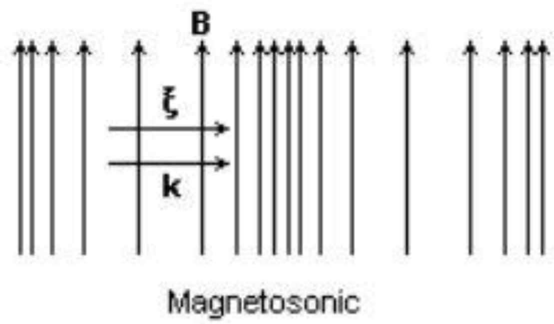
[01] Iwai, K; Shinya, K,; Takashi, K. and Moreau, R. (2003) "Pressure change accompanying Alfvén waves in a liquid metal" *Magnetohydrodynamics* 39(3): pp. 245-250, page 24

The perturbation equation gives a matrix,

$\overleftrightarrow{M}.\delta\vec{X} = 0 \quad \rightarrow \quad DET\left(\overleftrightarrow{M}\right) = 0 \quad \rightarrow$ Dispersion relation

Magneto-hydrodynamic waves



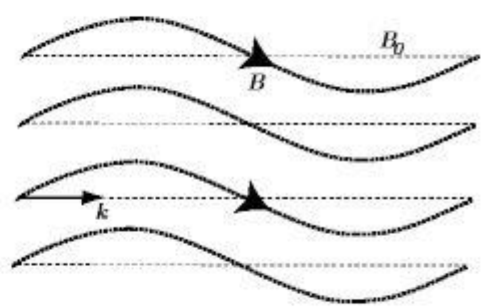


Magnetosonic wave

Magnetosonic waves propagating orthogonal to B_0. The displacement ξ is parallel to k but orthogonal to B_0. These are longitudinal waves.

Shear Alfvén waves

Shear Alfvén waves propagating parallel to B_0 and no propagation orthogonal to B_0. These are transverse waves.



The dispersion relationship,

$$\omega^2 = k_\parallel^2 V_A^2$$

The restoring force for the Shear Alfvén waves is magnetic tension. The displacement ξ is orthogonal to B_0 and k.

$\xi = \xi_x \hat{x}$, $B_0 = B_0 \hat{z}$ and $k = k_\perp \hat{y} + k_\parallel \hat{z}$

Shear Alfvén waves are incompressible. Since $k.\xi = 0$, the linearized continuity and energy equations show that both p_1 and ρ_1 are Zero.