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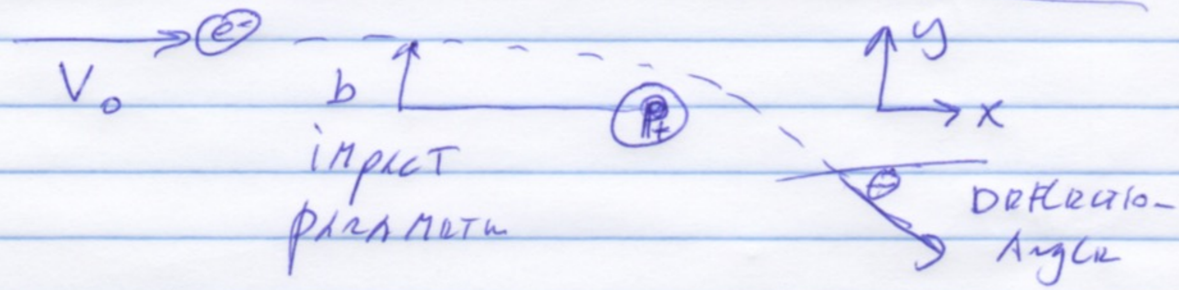
# Plasma & MHD Intro 1/31/17

## Coulomb Collision and Rutherford Cross-Section




**Task**: Calculate deflection angle $\theta$ as a function of impact parameter $b$ and impact velocity $V_0$

Specific assumptions here: e-p collision, center of gravity in proton

$$\boxed{m_e \frac{d\vec{V_e}}{dt} = \frac{-e^2}{4\pi\epsilon_0} \frac{\hat{r}}{r^2}}$$ ; momentum eqn. for electron

~~Switch to cylindrical coordinates~~

Note: This is the Newton problem of two bodies moving under gravitational attraction! Hence, we already know

1: All motions in plane $(x, y)$, $z = 0$

2: Angular momentum conserved; $L_z = bV_0 = r^2\dot{\varphi}$ (in cylindrical coordinates)

3: Energy conserved:

$$\frac{m_e}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - \frac{e^2}{4\pi\epsilon_0 r} = \underbrace{\frac{m_e}{2} V_0^2}_{\text{initial energy}}$$

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EQS OF MOTION IN CYLINDRICAL COORDINATES:

$$\boxed{\begin{aligned} m_e(\ddot{r} - r\dot{\varphi}^2) &= -\frac{e^2}{4\pi\varepsilon_0 r^2} \\ &\uparrow \\ &\text{CENTRIFUGAL FORCE} \\ m_e \frac{d}{dt}(r^2\dot{\varphi}) &= 0 \;;\ \text{NO TORQUE, CONSERVATION OF ANGULAR MOMENTUM} \end{aligned}}$$

SOLVE FOR $\dot{r}$ AND $\dot{\varphi}$ USING CONSERVATION EQS.

$$\frac{dr}{dt} = -\left\{ V_0^2 + \frac{2e^2}{4\pi\varepsilon_0 r m_e} - \frac{b^2 V_0^2}{r^2} \right\}^{1/2}$$

→ MINUS SIGN ON FIRST PART OF TRAJECTORY

$$\frac{d\varphi}{dt} = \frac{b V_0}{r^2}$$

ELIMINATE $dt$ (AND HENCE $t$, WHICH DOES NOT APPEAR EXPLICITLY)

$$d\varphi = \frac{-dr/r^2}{\left\{ \frac{1}{b^2} + \frac{e^2}{2\pi\varepsilon_0 m_e b^2 V_0^2 r} - \frac{1}{r^2} \right\}^{1/2}}$$

SUBSTITUTE; $y = b/r \quad \left(dy = -\frac{b}{r^2}\,dr\right)$

$Q \triangleq e^2/(4\pi\varepsilon_0 m_e V_0^2 b)$

$$\boxed{d\varphi = dy / \{1 - 2Qy - y^2\}^{1/2}}$$

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BOUNDARY CONDITIONS: $y = b/r$

$t = 0,\ r = \infty \Rightarrow y_0 = 0$

$r = r_{min}$ (CLOSEST APPROACH) $\Rightarrow y = y_{max}$ ($t = t_{max}$)

$$\varphi_{max} - \varphi_0 = \int_0^{y_{max}} dy \left\{ 1 - 2Qy + y^2 \right\}^{-1/2}$$

NOTE $y = y_{max}$ WHEN $\frac{dy}{d\varphi} = 0 \Rightarrow \underline{1 - 2Qy_M - y_M^2 = 0}$

SUBSTITUTE AGAIN: $\sin\Psi = \dfrac{Q + y}{(1+Q^2)^{1/2}}$

$\Rightarrow d\sin\Psi = \cos\Psi\, d\Psi = dy/(1+Q^2)^{1/2}$

$$\varphi_{max} - \varphi_0 = \int_0^{y_{max}} dy \left\{ 1 + Q^2 - (Q+y)^2 \right\}^{-1/2}$$

$$\boxed{\varphi_{max} - \varphi_0 = \int_{\Psi_0}^{\Psi_{max}} d\Psi = \Psi_{max} - \Psi_0}$$ SOLUTION

BOUNDARY CONDITIONS: 1) $y = 0 \Rightarrow \sin\Psi_0 = \dfrac{Q}{(1+Q^2)^{1/2}}$

2) $y = y_{max} \Rightarrow \sin\Psi_M = \dfrac{Q + y_{max}}{(1+Q^2)^{1/2}}$

$1 - 2Qy_M - y_M^2 = 0 \Rightarrow 1 + Q^2 - (Q + y_M)^2 = 0$ $\Bigg\}$ $\underline{\sin\Psi_M = 1}$ $\boxed{\Psi_M = \pi/2}$

AND, $\sin\Psi_0 = \dfrac{Q}{(1+Q^2)^{1/2}} \Rightarrow \boxed{\Psi_0 = \arctan Q}$

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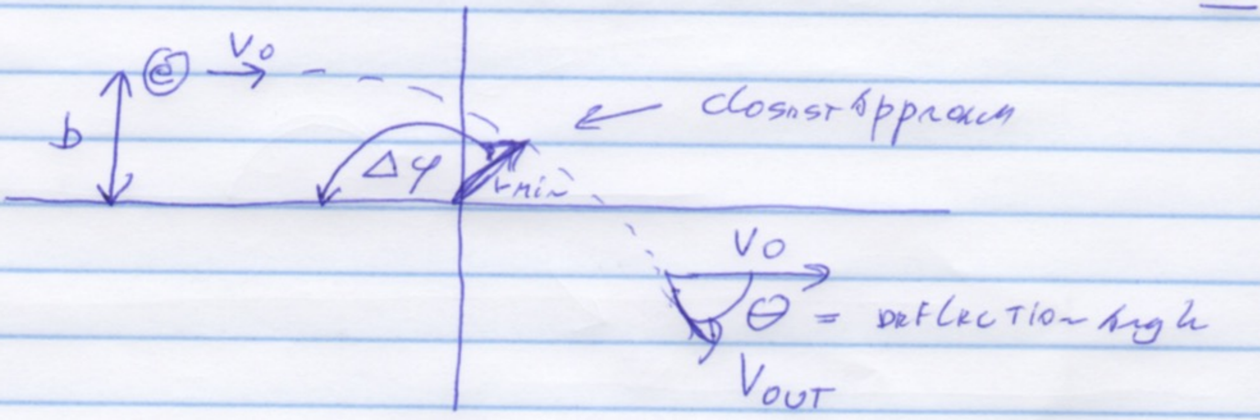
Hence $\boxed{\Delta\varphi = \frac{\pi}{2} - \arctan Q}$ Very elegant!

$\Delta\varphi$ = the change in position angle of the electron between $r_0 = \infty$ and closest approach, $r = r_{min}$

Does that make sense? $\left.\begin{array}{l} e = 0 \\ V_0, b = \infty \end{array}\right\}$ $\Delta\varphi = \pi/2$, straight line, no deflection

ok




Deflection angle $\theta$:

$$\boxed{\theta = \pi - 2\Delta\varphi = 2\arctan Q}$$

$$\boxed{Q = \frac{-e^2}{m_e V_0^2 b}}$$

Extremely elegant!