Plasma Physics Notes

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1 Mathematical Derivation of the DeBye Length

Starting with the Laplacian of the potentail we set that equal to the charge from the "cloud" and the central charge.

$$\varepsilon \nabla^2 \Phi = \frac{8\pi e^2 n_e}{kT^2}\Phi + Q\delta(\mathbf{r}) \tag{1}$$

The first time on the right can be found from the $\nabla \cdot \mathbf{E}$ and the series expansion of the charge distribution, this is the cloud term (screening); this term is porportional to $\frac{1}{\lambda_D^2}$. The second term comes from the charge at the center of the cloud.

The potential then shows this charge is effectively screened from other charges outside the DeBye length.

$$\Phi = \frac{Q}{4\pi r} e^{\frac{-\mathbf{r}}{\lambda_D}} \tag{2}$$

2 From Maxwells Equations to MHD cont.

$$\frac{\partial \mathbf{B}}{\partial t} = c(-\nabla \times \mathbf{E}) \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4}$$

$$\nabla \cdot \mathbf{D} = \varepsilon_0 \nabla \cdot \mathbf{E} = \rho \tag{5}$$

$$\frac{\partial \mathbf{D}}{\partial t} = c(\nabla \times \mathbf{H}) - \mathbf{j} \tag{6}$$

Note: ρ is screened out assuming small DeBye lengths and since $v/c << 1$, $\frac{\partial \mathbf{D}}{\partial t} \approx 0$

To further reduce the number of variables in these equations and to get a better understanding of a plasma we look at the force applied to both the electron and proton in the plasma.

Assume $n_e = n_p = n$

$$nm_e \frac{d\mathbf{v_e}}{dt} = -en(\mathbf{E} + \mathbf{v_e} \times \mathbf{B}) - \nabla \cdot \widehat{\mathbf{P_e}} - nm_e(\mathbf{v_e} - \mathbf{v_p})\nu_{e,p} \tag{7}$$

$$nm_p \frac{d\mathbf{v_p}}{dt} = en(\mathbf{E} + \mathbf{v_p} \times \mathbf{B}) - \nabla \cdot \widehat{\mathbf{P_p}} + nm_e(\mathbf{v_e} - \mathbf{v_p})\nu_{e,p} \tag{8}$$

Again, making some assumptions, $d\mathbf{v_e}/dt$ is close to zero so the left hand of the first equation is zero; and the gradients of the Pressure tensors are ignored at this point.

The other equation we need to remember is:

$$\mathbf{j} = ne(\mathbf{v_p} - \mathbf{v_e}) \tag{9}$$

Substituting the above back into the force equations we are left with:

$$0 = -en(\mathbf{E} + \mathbf{v_e} \times \mathbf{B}) + \frac{m_e \mathbf{j} \nu_{e,p}}{e} \tag{10}$$

$$nm_p \frac{d\mathbf{v_p}}{dt} = en(\mathbf{E} + \mathbf{v_p} \times \mathbf{B}) - \frac{m_e \mathbf{j} \nu_{e,p}}{e} \tag{11}$$

2.1 Plasma Velocity

As an aside the plasma velocity is the sum of the momenta of all particles in the plasma divided by their total mass. Again under the above assumptions then

$$\mathbf{v} = \frac{nm_e \mathbf{v_e} + nm_p \mathbf{v_p}}{nm_e + nm_p} \tag{12}$$

since the proton density and electron desity are the same this density can be cancelled leaving

$$\mathbf{v} = \frac{m_e \mathbf{v_e} + m_p \mathbf{v_p}}{m_e + m_p} \tag{13}$$

and if we assume $\frac{\mathbf{v_e}}{\mathbf{v_p}} < \frac{m_p}{m_e}$ then $\mathbf{v} \approx \mathbf{v_p}$. This is the case when there are collisons, if we assume no collision then $\mathbf{v_e}/\mathbf{v_p}$ could be on the order of the mass ratio and the plasma velocity would be different from the proton velocity.

Continuing on looking only at Equation (10) for the force on the electron. If we add and subtract a $en\mathbf{v_p} \times \mathbf{B}$ we can re-write the equation like this.

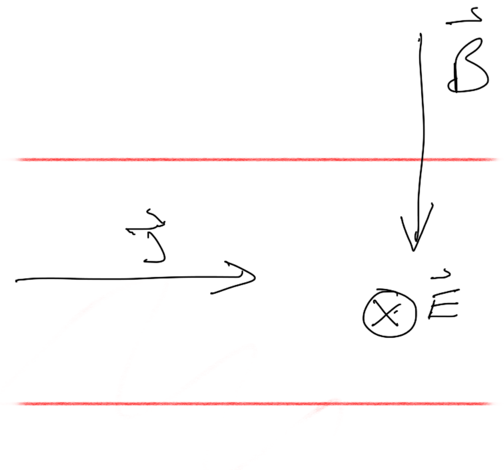
$$en[\mathbf{E} + (\mathbf{v_e} - \mathbf{v_p}) \times \mathbf{B}] + en\mathbf{v_p} \times \mathbf{B} = en\eta\mathbf{j} \tag{14}$$

where $\eta = \frac{m_e \nu_{e,p}}{ne^2}$.

Substituting for $\mathbf{j}$ and the proton veolcity for the plasma velocity this becomes

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta\mathbf{j} + \frac{\mathbf{j} \times \mathbf{B}}{ne} \tag{15}$$

This means we can eliminate $\mathbf{E}$ from our Maxwell Equations. the second term on the right is called the Hall term as it generates the perpendicular potential in a current flow in a magnetic field.



Maxwell's equations now become

$$\frac{\partial \mathbf{B}}{\partial t} = c\left(-\nabla \times \left[-\mathbf{v} \times \mathbf{B} + \eta \mathbf{j} + \frac{\mathbf{j} \times \mathbf{B}}{ne}\right]\right) (\text{+ previously ignored terms}) \tag{16}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{17}$$

$$c(\nabla \times \mathbf{B}) = \mu \mathbf{j} \tag{18}$$

3 Plasma Momentum

Now that an expression has been found for E we can continue to find the force on the plasma and therefore the momentum of the Plasma. Adding equation 10 to equation 11 we are left with

$$nm_p \frac{d\mathbf{v_p}}{dt} = -en(\mathbf{v_e} \times \mathbf{B}) + en(\mathbf{v_p} \times \mathbf{B}) \tag{19}$$

which $\mathbf{j}$ and $\mathbf{v} \approx \mathbf{v_p}$ can be subsituted into once again to give us

$$nm_p \frac{d\mathbf{v}}{dt} = \mathbf{j} \times \mathbf{B} \tag{20}$$

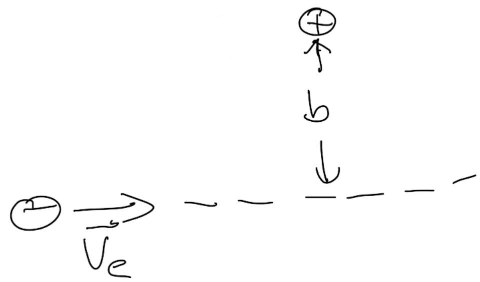
This equation states the you can either generate a force on the plasma by running a current through it, or by forcing motion on your plasma you can generate a dynamo.

Up until this point all velocities referenced have been averages for the particle type.

4 Collision Frequency, $\nu_{e,p}$

Given the figure below

In order to understand how the electron pass the proton we write out the equation of motion for the electron



$$m_e \frac{d\mathbf{v_e}}{dt} = \frac{-e^2}{4\pi r^2}\hat{\mathbf{r}} \tag{21}$$

because we know the plasma cloud shields the electrons we can then predict that if $r > \lambda_D$ that there will be minimal reaction. We can also take into account the energy.

$$\frac{1}{2} m_e {\mathbf{v_e}}^2 = \frac{e^2}{4\pi r} \tag{22}$$

If the kinetic energy is much larger than the potential energy then the proton would have very little effect, otherwise the electron motion would be changed a lot.