Plasma Ph120 3/10/2021

Sound Waves in Stratisfied Atmosphere

(1) Zero'th Order Atmosphere

$$\frac{\partial P}{\partial z} = -\rho g \hat{z} \qquad P = 2\rho R T = C_s^2 \rho$$

$$\Rightarrow \frac{1}{P}\frac{\partial P}{\partial z} = -\frac{g}{C_s^2} = -\frac{g}{2RT}$$

$$P(z) = P(z=0)\, \exp\left[\int_0^z -\frac{g}{2RT}\right]$$

$T(z)$ prescribed/measured

Take $T(z) = \text{constant} = T_0$; $H \hat{=} \frac{2RT_0}{g}$

H is the scale height $\sim$ 10 km in Earth's Atmosphere

Hence : $P(z) = P(z=0)\, e^{-z/H}$

(2) Looking for waves and stability, First order expansion, $\vec{V}_0 = \vec{0}$, ρ_0 and P_0 above

Assume $T = \text{constant} = T_0$

- T_1 defined by adiabatic energy law (no energy exchange) : $P = C\rho^\gamma$, γ is the adiabatic exponent

$$\underset{\text{isothermal}}{1} \le \gamma \le \underset{\text{adiabatic}}{5/3}$$

$\Rightarrow$ First order : $\boxed{P_1 = \gamma C_s^2 \rho_1}$

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⇒ SEARCHING FOR GRAVITY WAVES

(3) FIRST ORDER, NO MAGNETIC FIELD

$$\frac{\partial \rho_1}{\partial t} + (\vec{V}_1 \cdot \vec{\nabla})\rho_0 + \rho_0 \vec{\nabla} \cdot \vec{V}_1 = 0$$ MASS CONSERVATION

$$\rho_0 \frac{\partial \vec{V}_1}{\partial t} = -\vec{\nabla} P_1 - \rho_1 g \hat{z}$$ MOMENTUM CONSERV.

$(P_1 = \gamma \rho_1 \ ; \ T_0 = \text{CONSTANT})$

LOOKING FOR SOLUTIONS OF THE FORM:

$$\vec{V}_1(\vec{r},t), \rho_1(\vec{r},t) = \vec{V}_1', \rho' e^{-i(\omega t - \vec{k}\cdot\vec{r})}$$

$\vec{V}_1'$ AND ρ' ARE CONSTANTS (AMPLITUDE)

BECAUSE OF SYMMETRY AROUND THE $\hat{z}$-AXIS WE MAY SIMPLIFY : $\vec{k} = (k_x, 0, k_z)$

(4) BOOK KEEPING : ELIMINATE P_1 AND ρ_1 BY TAKING THE TIME DERIVATIVE OF THE MOMENTUM EQUATION AND SUBSTITUTION

$$\frac{\partial^2 \vec{V}_1}{\partial t^2} = C_s^2 \vec{\nabla}(\vec{\nabla}\cdot\vec{V}_1) - (\gamma-1) g \hat{z} (\vec{\nabla}\cdot\vec{V}_1) - g \vec{\nabla} V_{1,z}$$

$$\Rightarrow \omega^2 \vec{V}_1' = C_s^2 \vec{k}(\vec{k}\cdot\vec{V}_1') + i(\gamma-1) g \hat{z} (\vec{k}\cdot\vec{V}_1') + i g \vec{k} V_{1,z}'$$

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Repeat:

$$\omega^2 \vec{V}_1' = c_s^2 \vec{k}(\vec{k}\cdot\vec{V}_1') + i(\gamma-1)g\hat{z}(\vec{k}\cdot\vec{V}_1') + ig\vec{k}\,V_{1,z}'$$

(5) Solutions

A) Sound waves, pure, $g=0$

$$\omega^2 \vec{V}_1' = c_s^2 \vec{k}(\vec{k}\cdot\vec{V}_1')$$

Compressional, i.e. $(\vec{k}\cdot\vec{V}_1') \neq 0 \Rightarrow \omega^2 = k^2 c_s^2$

B) $g \neq 0$: Both acoustic and gravity waves

Book keeping: Take dot product of equation above with $\vec{k}$, and consider $\hat{z}$ component of same equations

a) $\omega^2(\vec{k}\cdot\vec{V}_1') = k^2 c_s^2 (\vec{k}\cdot\vec{V}_1') + i(\gamma-1)g k_z (\vec{k}\cdot\vec{V}_1') + \ldots$
$\ldots + i g k^2 V_{1,z}'$

b) $\omega^2 V_{1,z}' = k_z c_s^2 (\vec{k}\cdot\vec{V}_1') + i(\gamma-1)g(\vec{k}\cdot\vec{V}_1') + i g k_z V_{1,z}'$

a) + b) is a linear system with two unknowns. $V_{1,z}'$ and $(\vec{k}\cdot\vec{V}_1')$, of the form

$$\overleftrightarrow{A}\cdot\begin{pmatrix}\vec{k}\cdot\vec{V}_1' \\ V_{1,z}'\end{pmatrix} = 0. \quad \underline{\text{Hence } \mathrm{Det}(\overleftrightarrow{A}) = 0}$$

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In Search of gravity waves

$\mathrm{Det}(\overleftrightarrow{\chi}) = 0 \implies$

$$\omega^2(\omega^2 - N_s^2) - (\omega^2 - N^2 \sin^2\theta_g) k'^2 c_s^2 = 0$$

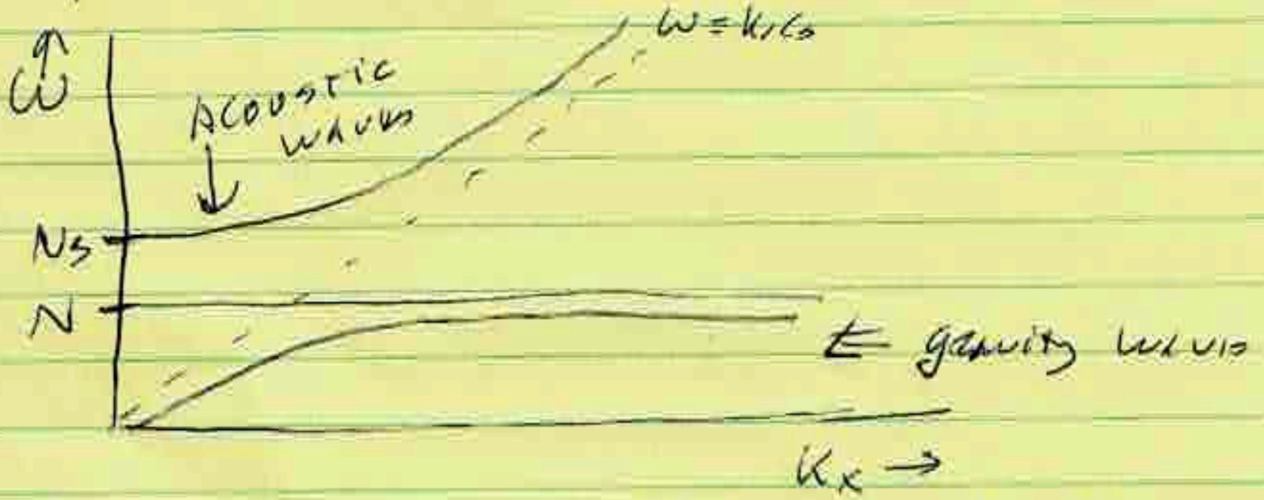
Dispersion relation, quadratic in $\omega^2 \Rightarrow$ solvable

$$\sin^2\theta \doteq 1 - k_z'^2/k'^2 \ ; \ \vec{k}' = \vec{k} - \frac{i\hat{z}}{2k}$$

$$N_s = \frac{\gamma g}{2c_s} \ ; \ N = \frac{g(\gamma-1)^{1/2}}{c_s}$$, Brunt-Vaisala Frequency

Note: gravity wave $\neq$ gravitational wave

Dispersion



- Gravity waves $\omega \ll k'c_s \rightarrow \omega = N\sin\theta_g$
- Sound waves $\omega \gg N \rightarrow \omega = k'c_s$
- Vertical, $k' = (0,0,k_z') \rightarrow \omega^2 = N_s^2 + k'^2 c_s^2$

cut off!