However, λ is assumed to be a constant, whereas r is a variable. Hence the above relationship cannot be true for all values of r. Hence the field cannot be modeled, over the entire space, as a linear force-free field.

The fact is that any linear force-free field cannot become 'smooth' at large distances. The magnetic field gradients tend to have the same length-scale over all space. This is obviously a severe handicap in many cases, such as the one we have just discussed.

13.3 Examples of linear force-free fields

In order to understand some properties of force-free fields in general, and some of the limitations of linear force-free fields in particular, it is helpful to consider some special examples. The first that we consider can be constructed in rectangular coordinates.

Consider a field configuration that is uniform in the y dimension, and for which each component is sinusoidal in the x-direction with wave number k. On examining (13.2.7), it appears that the field can vary exponentially in z, for example as e^{-lz}, if λ satisfies the condition $\lambda^2 < k^2$. On the other hand, if $\lambda^2 > k^2$, we could construct a solution by allowing for sinusoidal variation in the z-direction also, with wave number $(\lambda^2 - k^2)^{1/2}$.

If we are attempting to model the magnetic field of a solar active region, for instance, we will seek field configurations that extend throughout the half space $z > 0$, and decrease in strength with z. For this purpose, we must restrict our attention to values of λ that are, in magnitude, less than the magnitude of k. With this assumption, we may seek solutions of the form

$$\left.\begin{aligned} B_x &= B_{x,0}\sin(kx)e^{-lz}, \\ B_y &= B_{y,0}\sin(kx)e^{-lz}, \\ B_z &= B_0\cos(kx)e^{-lz}. \end{aligned}\right\} \qquad (13.3.1)$$

The components of $\nabla \times \mathbf{B}$ are found to be

$$\left.\begin{aligned} (\nabla\times\mathbf{B})_x &= lB_{y,0}\sin(kx)e^{-lz}, \\ (\nabla\times\mathbf{B})_y &= (-lB_{x,0} + kB_0)\sin(kx)e^{-lz}, \\ (\nabla\times\mathbf{B})_z &= kB_{y,0}\cos(kx)e^{-lz}. \end{aligned}\right\} \qquad (13.3.2)$$

On using (13.2.6), we now obtain the following three equations

$$\left.\begin{aligned} lB_{y,0} &= \lambda B_{x,0}, \\ -lB_{x,0} + kB_0 &= \lambda B_{y,0}, \\ kB_{y,0} &= \lambda B_0. \end{aligned}\right\} \qquad (13.3.3)$$

Hence the field (13.3.1) may be expressed as

$$\left.\begin{aligned} B_x &= k^{-1} l B_0 \sin(kx) e^{-lz}, \\ B_y &= k^{-1} \lambda B_0 \sin(kx) e^{-lz}, \\ B_z &= B_0 \cos(kx) e^{-lz}, \end{aligned}\right\} \quad (13.3.4)$$

where k, l and λ must be related by

$$l^2 = k^2 - \lambda^2. \quad (13.3.5)$$

This is the relationship that we would expect on the basis of the Helmholtz equation (13.2.7).

We see from (13.3.4) that the projection of magnetic field lines on the x–y plane are all parallel straight lines defined by

$$B_y = \frac{\lambda}{(k^2 - \lambda^2)^{1/2}} B_x. \quad (13.3.6)$$

The projections of magnetic field lines on the x–z plane are as shown in Fig. 13.3(*a*). A physical interpretation of this model is that we begin with a potential field of the form shown in (13.3.4) with $\lambda = 0$. We then shear the x–y plane in such a way that strips parallel to the y-axis are displaced in the y-direction. This shear is uniform, as shown in Fig. 13.3(*b*). If θ is the angle of inclination of field lines to the x-axis, we see from (13.3.6) that

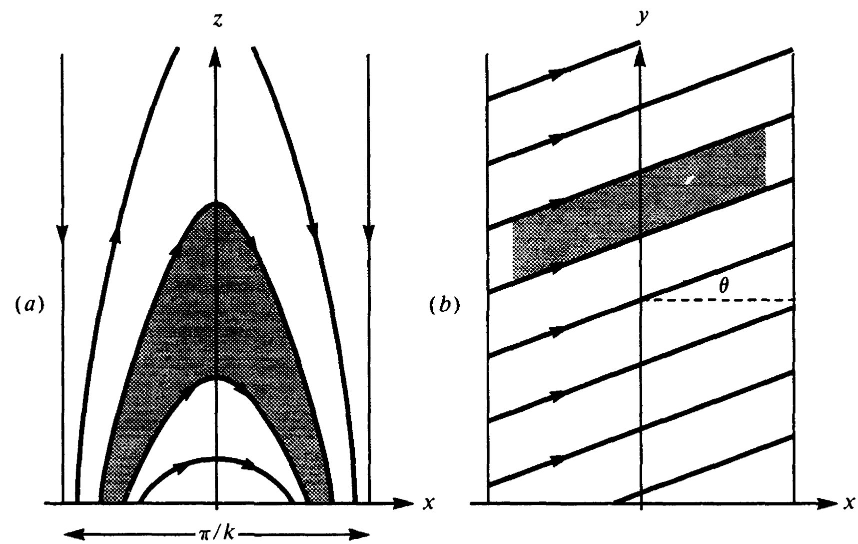

Fig. 13.3. Panel (a) shows the projection in the x–z plane of the force-free field defined by (13.3.4). Panel (b) shows the projection in the x–y plane. (Figure reproduced with kind permission from Priest 1982.)

$$\sin\theta = \lambda/k. \tag{13.3.7}$$

Hence increasing the shear (increasing θ) amounts to increasing λ. However this equation also shows that there is a limiting value λ: it cannot increase beyond the value $\lambda = k$. At this point, the field lines have the form

$$\left.\begin{aligned} B_x &= 0, \\ B_y &= B_0 \sin(kx), \\ B_z &= B_0 \cos(kx). \end{aligned}\right\} \tag{13.3.8}$$

This describes a magnetic-field configuration of constant strength, for which the direction of the magnetic-field vector rotates steadily with increasing x.

In this model, field lines extend to greater and greater height as the shear increases. The limiting case $\theta = \pi/2$ corresponds to infinite shear. In this case, all field lines extend to infinite height. On the other hand, there is no change in the horizontal magnetic-field configuration as shear increases. This is due to the fact that we have assumed an infinite array of line dipoles. We shall see, in a later section, that if we consider only a single line dipole, increasing shear causes field lines to expand horizontally as well as vertically.

13.4 The generating-function method

We again consider a magnetic-field configuration that has translational symmetry. (A similar analysis could be derived for a configuration of rotational (i.e. cylindrical) symmetry.) In terms of rectangular coordinates x, y, z, we assume that the field is uniform in the z-direction. Since $\nabla \cdot \mathbf{B} = 0$, we see that the magnetic field may be expressed as

$$\mathbf{B} = \left(\frac{\partial A}{\partial y},\ -\frac{\partial A}{\partial x},\ B_z \right). \tag{13.4.1}$$

In the following calculation, it is instructive to allow for a possible gas pressure in order to see the relationship between magnetic stress and plasma pressure. We therefore consider that the following force equation is satisfied

$$\frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p = 0. \tag{13.4.2}$$

On examining the three components of (13.4.2), we find that, since no quantity depends upon the variable z, the third component becomes

$$\frac{\partial B_z}{\partial x}\frac{\partial A}{\partial y} - \frac{\partial B_z}{\partial y}\frac{\partial A}{\partial x} = 0. \tag{13.4.3}$$